PROCEEDINGS

OF THE

AMERICAN SOCIETY OF CIVIL ENGINEERS

VOL. 60 DECEMBER, 1934 NO. 10

TECHNICAL PAPERS

AND

DISCUSSIONS

Published monthly, except June and July, at 99–129 North Broadway, Albany, N. Y., by the American Society of Civil Engineers, Editorial and General Offices at 33 West Thirty-ninth Street, New York, N. Y. Reprints from this publication may be made on condition that the full title of Paper, name of Author, page reference, and date of publication by the Society, are given.

Entered as Second-Class Matter, December 28, 1931, at the Post Office at Albany, N. Y., under the Act of March 3, 1879. Acceptance for mailing at special rate of postage provided for in Section 1103, Act of October 3, 1917, authorized on July 5, 1918.

Subscription (if entered before January 1) $8.00 per annum. Price $1.00 per copy.

Copyright, 1934, by the AMERICAN SOCIETY OF CIVIL ENGINEERS
Printed in the United States of America

CURRENT PAPERS AND DISCUSSIONS

		Discussion closes
Progress Report of Special Committee on Earths and Foundations	May, 1933	
Discussion	Aug., Sept., Oct., Nov., Dec., 1933, Jan., Aug., 1934	Closed
Water Power Development of the St. Lawrence River. *Daniel W. Mead*	Aug., 1933	
Discussion	Aug., Nov., Dec., 1933	Closed
Water-Bearing Members of Articulated Buttress Dams. *Hakan D. Birke*	Sept., 1933	
Discussion	Feb., 1934	Closed
Model of Calderwood Arch Dam. *A. V. Karpov and R. L. Templin*	Dec., 1933	
Discussion	Apr., May, Sept., 1934	Closed
An Approach to Determinate Stream Flow. *Merrill M. Bernard*	Jan., 1934	
Discussion	Mar., Apr., May, Sept., Nov., 1934	Closed
Discharge Formula and Tables for Sharp-Crested Suppressed Weirs. *C. G. Cline* (Author's closure)	Jan., 1934	
Discussion	May, Sept., Dec., 1934	Closed
Renewal of Miter-Gate Bearings, Panama Canal. *Clinton Morse*	Jan., 1934	
Discussion	May, 1934	Closed
Williot Equations for Statically Indeterminate Structures in Combination with Moment Equations in Terms of Angular Displacements. *Charles A. Ellis*	Jan., 1934	
Discussion	Sept., 1934	Closed
Flexible "First Story" Construction for Earthquake Resistance. *Norman B. Green*	Feb., 1934	
Discussion	May, Aug., Nov.,	Dec., 193
Analysis of Sheet-Pile Bulkheads. *Paul Baumann*	Mar., 1934	
Discussion	May, Aug., Oct., Nov., Dec., 1934	Dec., 193
A Generalized Deflection Theory for Suspension Bridges. *D. B. Steinman*	Mar., 1934	
Discussion	May, Aug., Sept., Nov., Dec., 1934	Dec., 193
Sand Mixtures and Sand Movement in Fluvial Models. *Hans Kramer*	Apr., 1934	
Discussion	Aug., Sept., Nov., Dec., 1934	Dec., 193
Laboratory Tests of Multiple-Span Reinforced Concrete Arch Bridges. *Wilbur M. Wilson*	Apr., 1934	
Discussion	Aug., Sept., Dec., 1934	Dec. 193
The Reservoir as a Flood-Control Structure. *George R. Clemens*	May, 1934	
Discussion	Nov., Dec., 1934	Dec., 193
Stresses in Space Structures. *F. H. Constant*	May, 1934	
Discussion	Sept., Oct., 1934	Dec., 193
Experiments with Concrete in Torsion. *Paul Andersen*	May, 1934	
Discussion	Aug., Oct., Dec., 1934	Dec., 193
Wave Pressures on Sea-Walls and Breakwaters. *David A. Molitor*	May, 1934	
Discussion	Sept., Dec., 1934	Dec., 193
Flow of Water Around Bends in Pipes. *David L. Yarnell and Floyd A. Nagler*	Aug., 1934	
Discussion	Nov., 1934	Jan., 193
Street Thoroughfares: A Symposium	Aug., 1934	
Discussion	Oct., Nov., 1934	Jan., 193
Eccentric Riveted Connections. *Eugene A. Dubin*	Aug., 1934	
Discussion	Dec., 1934	Jan., 193
Determination of Trapezoidal Profiles for Retaining Walls. *A. J. Sutton Pippard*	Aug., 1934	
Discussion	Dec., 1934	Jan., 193
Security from Under-Seepage—Masonry Dams on Earth Foundations. *E. W. Lane*	Sept., 1934	
Discussion	Dec., 1934	Jan., 193
Uplift and Seepage Under Dams on Sand. *L. F. Harza*	Sept., 1934	
Discussion	Dec., 1934	Uncertain
An Asymmetric Probability Function. *J. J. Slade, Jr.*	Oct., 1934	
Discussion	Dec., 1934	Jan., 193
Analysis of Continuous Structures by Traversing the Elastic Curves. *Ralph W. Stewart*	Oct., 1934	
Discussion	Dec., 1934	Jan., 193
Relation Between Rainfall and Run-Off from Small Urban Areas. *W. W. Horner and F. L. Flynt*	Oct., 1934	Jan., 193
The Silt Problem. *J. C. Stevens*	Oct., 1934	Uncertain
Effect of Secondary Stresses Upon Ultimate Strength. *John I. Parcel and Eldred B. Murer*	Nov., 1934	Feb., 193
The Springwells Filtration Plant, Detroit, Michigan. *Eugene A. Hardin*	Nov., 1934	Feb., 193

NOTE.—The closing dates, herein published, are final except when names of prospective discusser's are registered for special extension of time.

CONTENTS FOR DECEMBER, 1934

PAPERS

PAGE

Analysis of Multiple Arches.
By Alexander Hrennikoff, Esq. 1403

Rational Design of Steel Columns.
By D. H. Young, Jun. Am. Soc. C. E. 1421

A Direct Method of Moment Distribution.
By T. Y. Lin, Jun. Am. Soc. C. E. 1451

REPORTS

Water Supply Engineering: Progress Report of the Committee of the Sanitary Engineering Division 1463

DISCUSSIONS

Discharge Formula and Tables for Sharp-Crested Suppressed Weirs.
By C. G. Cline, Esq. 1485

Analysis of Sheet-Pile Bulkheads.
By Theodor von Kármán, M. Am. Soc. C. E. 1486

Sand Mixtures and Sand Movement in Fluvial Models.
By Joseph B. Tiffany, Jr., Jun. Am. Soc. C. E., and Carl E. Bentzel, Esq. 1489

Laboratory Tests of Multiple-Span Reinforced Concrete Arch Bridges.
By M. Hirschthal, M. Am. Soc. C. E. 1496

The Reservoir as a Flood-Control Structure.
By Messrs. F. Knapp, and Charles S. Bennett. 1499

Experiments with Concrete in Torsion.
By A. W. Fischer, Esq. 1502

Wave Pressures on Sea-Walls and Breakwaters.
By Charles T. Leeds, M. Am. Soc. C. E. 1504

CONTENTS FOR DECEMBER, 1934 (*Continued*)

PAGE

Determination of Trapezoidal Profiles for Retaining Walls.
By Messrs. E. S. Lindley, and Kenneth L. DeBlois 1509

Analysis of Continuous Structures by Traversing the Elastic Curves.
By Messrs. Garrett B. Drummond, Austin H. Reeves, E. G. Paulet, Adolphus Mitchell, and David M. Wilson 1511

Eccentric Riveted Connections.
By Messrs. Carlton T. Bishop, James R. Bole, Albert Wertheimer, Kenneth L. DeBlois, Jonathan Jones, Armin Rozman, and A. E. R. de Jonge 1518

Uplift and Seepage Under Dams on Sand.
By Messrs. Edward Godfrey, and Hibbert M. Hill 1529

A Generalized Deflection Theory for Suspension Bridges.
By Messrs. Sterling Johnston, Harold E. Wessman, and C. H. Gronquist 1532

Security from Under-Seepage — Masonry Dams on Earth Foundations.
By Messrs. William P. Creager, and L. F. Harza 1538

For Index to all Papers, the discussion of which is current in PROCEEDINGS, *see page 2*

The Society is not responsible for any statement made or opinion expressed in its publications

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

PAPERS

ANALYSIS OF MULTIPLE ARCHES

BY ALEXANDER HRENNIKOFF[1], ESQ.

SYNOPSIS

A continuous system of arches on elastic piers is analyzed in this paper, and a method of finding moments and horizontal thrusts at the ends of individual arch spans and piers, is presented. The subject of designing multiple arches, in the wide sense of the term, "design," is not treated; the paper is restricted to the discussion of a problem in engineering mechanics of determining stresses under given conditions of loading when all dimensions of the structure are known.

The method is based on the well-known principle of moment distribution originated by Hardy Cross, M. Am. Soc. C. E.,[2] but the manner of applying that principle, and various details of the method are original.

OUTLINE OF THE METHOD

The proposed method may be divided into the following operations:

(*a*) Analyze the individual spans on the assumption that they are fixed-ended and, on this assumption, determine the end moments and thrusts for each arch.

(*b*) Find the unbalanced moments and thrusts at each pier-head by adding, algebraically, the fixed-ended moments and thrusts coming from two adjacent arch spans (referring to a specific example, it will be assumed in Fig. 1, that the unbalanced joint functions are present only at the pier-head, *C*, none being present at the joint, *B*).

(*c*) Determine a series of quantities that may be appropriately termed, "the end distribution factors;" they are similar to distribution factors and carry-over factors in the method of moment distribution, and their nature will be explained subsequently.

(*d*) Determine the "joint distribution factors," four in number, which represent moments and thrusts at the joint with unbalanced forces, *C*, when it

NOTE.—Discussion on this paper will be closed in March, 1935, *Proceedings*.

[1] Instructor, Dept. of Civ. Eng., Univ. of British Columbia, Vancouver, B. C., Canada.

[2] *Transactions*, Am. Soc. C. E., Vol. 96 (1932), p. 1.

is given a unit rotation or a unit horizontal displacement by some outside agency. These joint factors are equal to algebraic sums of the respective end distribution factors (Operation (*c*)) for the three members meeting at Joint *C*.

(*e*) Distribute the unbalanced moment and thrust at Point *C*. This requires finding the rotation and horizontal displacement of Joint *C*, necessary to balance the forces at the joint, an operation involving the solution of two simple equations, or the construction of a diagram.

(*f*) Determine the resultant forces at the ends of all members.

Referring to Operation (*c*), there are four end distribution factors at each end of each member when the joint, *C*, moves (by member is meant each individual arch span or pier, so that, in all, there are five members in Fig. 1): (1) The rotation moment factor, m_a; (2) the rotation thrust factor, h_a; (3) the displacement moment factor, m_Δ; and (4) the displacement thrust factor, h_Δ. The first two factors are, respectively, the moment and the thrust at the end of any member such as Point *A* in Fig. 1, when Joint *C*, with an unbalanced fixed-ended moment and thrust, is given a unit rotation without any linear displacement. Similarly, m_Δ and h_Δ are moments and thrusts that occur when Joint *C* is given a unit horizontal displacement, without any rotation or any vertical displacement.

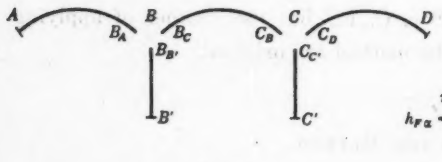

FIG. 1.

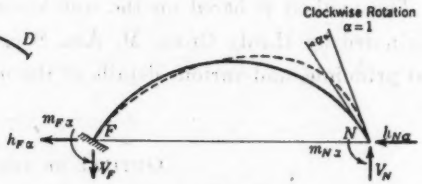

FIG. 2.

In the method of this paper unlike the method of moment distribution, the pier-head, *B*, adjacent to the one undergoing the movements, is not considered fixed, but is allowed to move its proper amount. This fact explains the difference in the method of determining distribution factors of two groups of members in Fig. 1. Members of the first group, *CC'* and *CD*, have one end fixed, the other end undergoing a known movement (unit rotation or unit translation). Their distribution factors will be found by formulas, derived subsequently. The other three members have unknown movements of one end, and their distribution factors will be found by means of a special algebaic process.

In concluding this brief outline it may be mentioned that the method suggested consists largely in following a certain simple arithmetical procedure, without recourse to higher mathematics. Formulas are used only in evaluating the end distribution factors of single members, and the constants involved are those commonly used in expressing the familiar elastic properties of arches and piers.

End Distribution Factors of Single Arches and Piers

Rotation Factors of a Symmetrical Arch Rib.—Let *FN* (Fig. 2) be a single symmetrical span of a multiple arch. It is required to find expressions for the moments and thrusts at the ends, *N* and *F*, when *F* is kept fixed, and *N* is made to rotate, without any linear displacement, through an angle, $\alpha = 1$ radian, in positive direction (which will be assumed to be clockwise).

The end, *N*, which moves, will be referred to as the "near" end, and the fixed end, *F*, will be termed the "far" end. Then, rotation thrust factors and rotation moment factors at the near end will be designated, respectively, $h_{N\alpha}$ and $m_{N\alpha}$, and similar quantities at the end, *F*, will be denoted by $h_{F\alpha}$ and $m_{F\alpha}$.

A definite agreement as to the exact meaning and signs of these symbols is most important. In line with a common convention in the method of moment distribution, it will be assumed that m and h are the moments and thrusts with which the arch acts on the joint; and the positive directions for these actions will be clockwise for the moment, and to the right for the thrust.

Since each joint acts on the arch with forces equal and opposite to those with which the arch acts on the joint, a free-body diagram of the arch with Joint *N* rotated clockwise through an angle, $\alpha = 1$ radian, will appear as shown in Fig. 2. Directions for arrows, m and h, in this diagram are determined by the aforementioned convention concerning meaning and signs of the end distribution factors. The actual forces may have directions opposite the arrows shown, in which case the corresponding distribution factors will be found to be negative. Thus, it is quite evident that a clockwise moment is required at the end, *N*, to effect clockwise rotation of this end; consequently, the moment factor, $m_{N\alpha}$, will ultimately be negative. In addition to m and h, vertical reactions, V_N and V_F, will be required to keep the span in static equilibrium.

The derivation of formulas for the moment factors, m, and the thrust factors, h, is not a special feature of this paper, but is given for completeness. The necessary expressions can be found easily by the neutral point method. Fig. 3 is drawn for the same conditions of deformation as Fig. 2. The arch

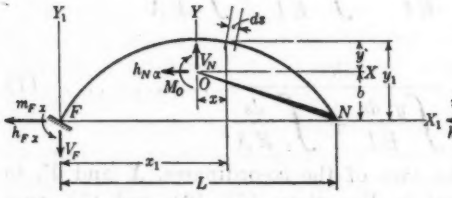

FIG. 3.

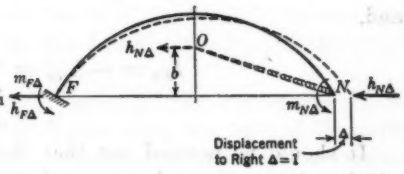

FIG. 4.

has a rigid arm extending from the end, *N*, to the neutral point, *O*, and is loaded at Point *O* with the forces, $h_{N\alpha}$, V_N, and M_O, which have been moved there from Point *N*. Since systems of forces in Figs. 2 and 3 are equivalent to each other, the following relation is evident from statics:

$$m_{N\alpha} = -\left[M_o + V_N \frac{L}{2} - h_{N\alpha}\, b \right] \quad \ldots\ldots\ldots\ldots\ldots (1)$$

By assumption, the only movement at Point N is rotation through an angle, $\alpha = 1$ (see Fig. 2); consequently, a composite movement occurs at the neutral point, O, as follows: (1) Clockwise rotation, $\alpha = 1$; (2) vertical displacement, $\frac{L}{2}$, upward; and (3) a horizontal displacement, b, to the right, in which, b is the height of the neutral point above the springing line.

In order to effect these displacements, the forces must be exerted at Point O such as to satisfy the following conditions (derived from the well-known properties of the neutral point):

$$M_o \int \frac{ds}{EI} = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$V_N \int \frac{x^2\,ds}{EI} = \frac{L}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

and,

$$-h_{Na}\left[\int \frac{y^2\,ds}{EI} + \int \frac{ds}{EA}\right] = b \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

The unknowns, M_o, V_N, and h_{Na}, are easily found from these relations; and, then, m_{Na}, m_{Fa}, and h_{Fa} are determined from Equation (1), and from the conditions of equilibrium of the arch. The resultant expressions for the rotation distribution factors of a single arch are:

$$m_{Na} = -\left[\frac{1}{\int \frac{ds}{EI}} + \frac{\left(\frac{L}{2}\right)^2}{\int \frac{x^2\,ds}{EI}} + \frac{b^2}{\int \frac{y^2\,ds}{EI} + \int \frac{ds}{EA}}\right] \quad \ldots\ldots (5)$$

$$m_{Fa} = \frac{1}{\int \frac{ds}{EI}} - \frac{\left(\frac{L}{2}\right)^2}{\int \frac{x^2\,ds}{EI}} + \frac{b^2}{\int \frac{y^2\,ds}{EI} + \int \frac{ds}{EA}} \quad \ldots\ldots (6)$$

and,

$$h_{Fa} = -h_{Na} = \frac{b}{\int \frac{y^2\,ds}{EI} + \int \frac{ds}{EA}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (7)$$

It should be pointed out that the axes of the co-ordinates, X and Y, to which values of x and y are referred in Equations (5), (6), and (7), pass through the neutral point, O (Fig. 3), and are directed horizontally and vertically. The distance, b, determining the location of the neutral point, is found from the relation:

$$b = \frac{\int \frac{y_1\,ds}{EI}}{\int \frac{ds}{EI}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

The following will be found useful in evaluating the integrals:

$$\int \frac{x^2\,ds}{EI} = \int \frac{x_1^2\,ds}{EI} - \left(\frac{L}{2}\right)^2 \int \frac{ds}{EI} \quad \ldots\ldots\ldots\ldots\ldots (9)$$

and,

$$\int \frac{y^2\,ds}{EI} = \int \frac{y_1^2\,ds}{EI} - b^2 \int \frac{ds}{EI} \quad \ldots\ldots\ldots\ldots\ldots (10)$$

In Equations (8), (9) and (10), x_1 and y_1 are co-ordinates of the arch axis referred to Axes X_1 and Y_1 with the origin, F, at the springing.

Displacement Factors of a Symmetrical Arch Rib.—Following the same convention as in the case of the rotation factors, the displacement factors are defined as the moments and horizontal thrusts with which the arch acts on the joints when the far end, F, remains fixed, and the near end, N, is displaced horizontally to the right (with no rotation) a distance of one unit of length (see Fig. 4). Positive directions for these actions of the arch on the joints (not of the joints on the arch) will be again assumed as to the right for thrusts, and clockwise for moments.

The movement of the neutral point under these circumstances will evidently be the same as that of the point, N, namely, one unit of length to the right; and this may be affected by a single horizontal force, $h_{N\Delta}$, applied at Point O, as shown by the broken line in Fig. 4, thus:

$$h_{N\Delta} = -\frac{1}{\int \frac{y^2\,ds}{EI} + \int \frac{ds}{EA}} \quad \ldots\ldots\ldots\ldots\ldots (11)$$

Expressions for the other factors will be found from the conditions of equilibrium:

$$h_{F\Delta} = -\,h_{N\Delta} \quad \ldots\ldots\ldots\ldots\ldots (12)$$

and,

$$m_{F\Delta} = -\,m_{N\Delta} = \frac{b}{\int \frac{y^2\,ds}{EI} + \int \frac{ds}{EA}} \quad \ldots\ldots\ldots\ldots\ldots (13)$$

Equations (5) to (13), inclusive, developed on the condition that the right end of the arch moves and the left end remains fixed, hold true in exactly the same form (and with the same signs) when the left end moves, and the right end stands still.

For investigating the influence of movements of the abutments, it is necessary to write expressions for terminal forces when one end of the arch settles vertically without rotation through a distance of one unit of length, and the other end remains fixed. It is easy to prove that no horizontal thrusts occur in this case, and that the end moments are equal to,

$$m_V = \frac{\frac{L}{2}}{\int \frac{x^2\,ds}{EI}} \quad \ldots\ldots\ldots\ldots\ldots (14)$$

the sign being plus on both ends if the right end of the arch moves down.

Rotation and Displacement Factors for a Pier with Fixed Base.—Fig 5 represents an elastic pier, fixed at the base, B, with a unit clockwise rotation at the top, T, with no linear displacement, and subjected to the action of the forces, m_{Ta} and h_{Ta} (reversed rotation factors for the pier). In Fig. 6 these

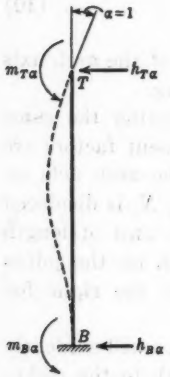

FIG. 5.

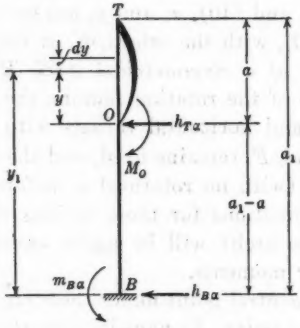

FIG. 6.

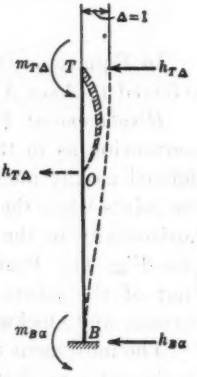

FIG. 7.

forces have been moved as M_o and h_{Ta} to the extremity of a rigid arm terminating at the neutral point of the pier, a distance, a, below its top. Evidently,

$$m_{Ta} = -[M_o + h_{Ta}\, a] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (15)$$

The neutral point, O, has the following movements: Clockwise rotation, $a = 1$, and linear displacement to the left, a. These are produced by,

$$h_{Ta} = \frac{a}{\int \frac{y^2\, dy}{E\, I}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (16)$$

and, $M_o = \dfrac{1}{\int \frac{dy}{E\, I}}$; then, from Equation (15),

$$m_{Ta} = -\left[\frac{1}{\int \frac{dy}{E\, I}} + \frac{a^2}{\int \frac{y^2\, dy}{E\, I}}\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots (17)$$

The y - co-ordinates of the pier axis, under the integral signs in Equations (16) and (17), are measured from the origin at the neutral point. In order to simplify the evaluation of the integrals, the following relation may be used:

$$\int \frac{y^2\, dy}{E\, I} = \int \frac{y_1^2\, dy_1}{E\, I} - (a_1 - a)^2 \int \frac{dy_1}{E\, I} \quad \ldots\ldots\ldots\ldots\ldots (18)$$

in which, y_1 and a_1 are measured from the base of the pier (see Fig. 6).

Similar argument leads to the following expressions for the displacement factors of a pier with fixed base (Fig. 7):

$$h_{T\Delta} = -\frac{1}{\int \frac{y^2\,dy}{E\,I}} \qquad (19)$$

and,

$$m_{T\Delta} = \frac{a}{\int \frac{y^2\,dy}{E\,I}} \qquad (20)$$

Rotation and Displacement Factors for a Pier That Can Move at the Base.—When investigating the influence of yielding foundations under a pier it is necessary to write expressions for the rotation and displacement factors of a pier that can rotate or move horizontally at the base, while the head remains fixed. Using the foregoing notation and convention as to the signs and directions of motions, expressions for thrusts and moments at the pier base (h_{Ba}, m_{Ba}, $h_{B\Delta}$, and $m_{B\Delta}$) can be obtained from corresponding expressions in Equations (16), (17), (19), and (20) at the pier-head by substituting $-(a_1 - a)$ for a. The force functions at the top of the pier, while the pier-head still remains locked, can then be found by statics.

Numerical Examples

Application of this method will be demonstrated by two examples of arch systems with ribs and piers of the proportions used by Charles S. Whitney,

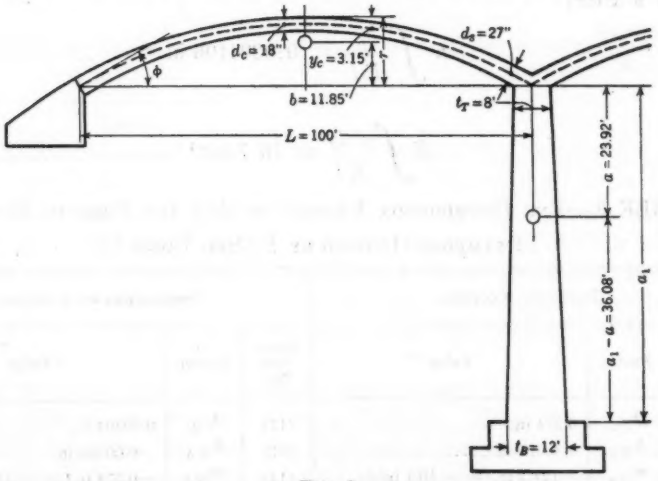

Fig. 8.

M. Am. Soc. C. E., in his fundamental papers on arch design.[3] All ribs and piers (see Fig. 8) will be assumed to be 1 ft thick, and will have the properties listed in Table 1.[4]

[3] *Transactions*, Am. Soc. C. E., Vol. 88 (1925), p. 931; and Vol. 90 (1927), p. 1094.
[4] *Loc. cit.*, Vol. 90 (1927), pp. 1112 and 1120.

TABLE 1.—PROPERTIES OF RIBS AND PIERS IN NUMERICAL EXAMPLES
(SEE FIG. 8)

Length of span, L, in inches	1 200	Distance, y_c, at the crown from the arch rib axis to the neutral point, in feet	3.15
Rise of arch axis, r, in feet	15	Thickness of Pier, in Feet:	
Ratio, w, of unit loads, w_s, at springing to w_c, at crown	4.70	(*a*) At the top, t_T	8.0
Ratio, n, of $\frac{I_c}{I_s \cos \phi}$	0.339	(*b*) At the bottom, t_B	12.0
Coefficient, k, in $W = \cosh k$	1.5	Moment of Inertia of Arch Barrel, in Inches[4]:	
Height of pier, a_1, in feet	60	(*a*) At the crown, I_c	6 399
Moment of inertia, I_T, of pier at the top $\left(= 1.1 \times \frac{8^3}{12}\right)$, in feet[4]	46.85	(*b*) At the springing, I_s	24 560
Thickness of Arch Barrel, in Inches:		Vertical distance, b, of neutral point above springing, in inches	142.2
(*a*) At the crown, d_c	18	Total height, $a_1 + b$, in feet	71.85
(*b*) At the springing, d_s	27	Distance, a, in inches (see Fig. 8)	287
Cosine ϕ	0.769	Distance, $a_1 - a$, in inches (see Fig. 8)	433

The following values have been found from the formulas developed by Mr. Whitney:

For an Arch Rib:

$$E \int \frac{ds}{EI} = 0.1252 \text{ in.}^{-3} \quad \ldots\ldots(21)$$

$$E \int \frac{x^2\, ds}{EI} = 11\,354 \text{ in.}^{-1} \quad \ldots\ldots(22)$$

and,

$$E \int \frac{y^2\, ds}{EI} = 247.9 \text{ in.}^{-1} \quad \ldots\ldots(23)$$

For a Pier:

$$E \int \frac{dy}{EI} = 0.0004106 \text{ in.}^{-3} \quad \ldots\ldots(24)$$

and,

$$E \int \frac{y^2\, dy}{EI} = 16.7 \text{ in.}^{-1} \quad \ldots\ldots(25)$$

TABLE 2.—END DISTRIBUTION FACTORS OF RIBS AND PIERS IN NUMERICAL EXAMPLES (DIVIDED BY E; SEE TABLE 1)

Rotation Factors			Displacement Factors		
Equation No.	Factor	Value	Equation No.	Factor	Value
(7)	h_{Fa}	0.574 in.^{2}	(12)	$h_{F\Delta}$	0.00404 in.
(7)	h_{Na}	—0.574 in.^{2}	(11)	$h_{N\Delta}$	—0.00404 in.
(5)	m_{Na}	—121.2 in.^{2} = — 10.1 in^2-ft	(13)	$m_{N\Delta}$	—0.574 in.^{2} = — 0.0478 in-ft.
(6)	m_{Fa}	57.8 in.^{2} = 4.82 in^2-ft	(13)	$m_{F\Delta}$	0.574 in.^{2} = 0.0478 in-ft.
(16)	h_{Ta}	17.19 in.^{2}	(19)	$h_{T\Delta}$	—0.0599 in.
(17)	m_{Ta}	—7371 in.^{2} = — 614.2 in^2-ft	(20)	$m_{T\Delta}$	17.19 in.^{2} = 1.433 in-ft.
(14)	m_s	0.0528 in^2 = 0.0044 in-ft	.....	.....	.....

The influence of rib-shortening, expressed by the integral, $\int \frac{ds}{EA}$, is slight and will be disregarded in the following examples. Numerical values for the end distribution factors are obtained by substituting the quantities in Equations (21) to (25) in Expressions (5) to (20). The results are listed in Table 2, the factors being divided by E. For example, the rotation factor, h_{Fa}, in Table 2, is actually $\frac{h_{Fa}}{E}$.

It is easy to understand that the absolute values of distribution factors are not essential, and that quantities proportional to them will serve as well in distributing unbalanced joint moments and thrusts. Moreover, the coefficient of proportionality for rotation factors may be made different from that for the displacement factors. A change in this coefficient may be advisable in order to raise or to lower the values of the factors, thus avoiding the inconvenience of using numbers that are too large or too small.

In the following numerical examples, the quantities, $\frac{h_a}{E}$ and $\frac{m_a}{E}$, will be used as the rotation factors, and the quantities, $\frac{100\,h_\Delta}{E}$ and $\frac{100\,m_\Delta}{E}$, as the displacement factors, $\frac{1}{E}$ and $\frac{100}{E}$ being the coefficients of proportionality.

It is evident from the physical meaning of the distribution factors that: m_a is measured in force length units; h_a, in force units; m_Δ, in $\frac{\text{force length}}{\text{length}}$ units; and, h_Δ, in $\frac{\text{force}}{\text{length}}$ units.

The important feature is that the moment factors have length dimensions of a power one degree higher than that of the corresponding thrust factors. The same characteristic is preserved in the units of the proportional values (see Table 2). This extra length dimension (an exponent of 3 in m_{Na}, for example, as compared with 2 in h_{Na}) is different from the remaining dimensions, and although the others may be expressed either in inches or in feet, the units of this particular dimension must agree with the length units of the fixed-ended moments.

This is the reason for the seemingly peculiar combinations of inches and feet in the units of the moment factors in Table 2. These combinations evidently result from the fact that the fixed-ended moments are expressed in kip-feet. Of course, it would be quite correct, although inconvenient because of small values, to have the proportional distribution factors expressed in feet only. Aside from this single qualification, the units of distribution factors are quite immaterial, and may be disregarded completely, unless the absolute values of terminal deformations are considered, as in the case of yielding foundations.

Example 1.—Table 3 contains the complete solution of a 2-span arch. Moments are expressed in kip-feet, and thrusts, in kips. The ribs and piers have the dimensions and elastic properties listed in Tables 1 and 2. The structure is loaded to the middle of the left span, as shown in Fig. 9, with a uniform load of $w = 0.1$ kip per lin ft. Pier Base B' and the abutments, A and C, are considered as absolutely fixed.

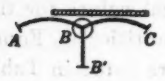

FIG. 9.—EXAMPLE 1.

TABLE 3.—END MOMENTS AND THRUSTS, TWO-SPAN ARCH BRIDGE
(SEE FIG. 9)
(Thrusts, h, are in kips; and moments, m, are in kip-feet)

Item No.	Conditions	End Moments and Thrusts for Member: A_B	Joint B								C_B
			B_A		BB'		B_C		Total, Joint B		
		m*	h	m	h	m	h	m	h (Columns (3), (5), and (7)	m (Columns (4), (6), and (8)	m†
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
1	$\alpha = \frac{1}{E}$	+ 4.82	−0.574	−10.10	+17.19	−614.2	−0.574	−10.10	+16.04	−634.4	+4.82
2	$\Delta = \frac{100}{E}$	+ 4.78	−0.404	− 4.78	− 5.99	−143.3	−0.404	− 4.78	− 6.80	+133.7	+4.78
3	Fixed-ended forces....	−25.02	+4.482	−12.40	.....	.....	−8.964	−12.62	−4.482	−25.02	+12.62
4	$\alpha = \frac{0.3541}{E}$ ..	− 1.71	+0.203	+ 3.58	− 6.09	+217.5	+0.203	+ 3.58	.....	.....	− 1.71
4	$\Delta = \frac{149.4}{E}$ ..	− 7.15	+0.603	+ 7.15	+ 8.95	−214.0	+0.603	+ 7.15	.....	.....	− 7.15
6	Two-span forces....	−33.88	+5.288	− 1.67	+ 2.87	+ 3.56	−8.158	− 1.89	0.0	0.0	+ 3.76

* Values of h are the same as in Column (3). † Value of h are the same as in Column (7).

The vertical columns in Table 3 contain moments and thrusts. The values refer to the ends of the various members, thus: Columns (3) and (4) (Table 3), headed "B_A", give forces at End B of Member BA, Fig. 9. Columns (9) and (10) contain the values of forces acting on Joint B. They are computed by adding, algebraically, the quantities of the three preceding columns, B_A, B_B', and B_C.

The conditions or causes that produce the forces are listed in Column (1), Table 3. Items Nos. 1 and 2 contain forces caused by the rotation, $\alpha = \frac{1}{E}$, and the displacement, $\Delta = \frac{100}{E}$, of Joint B; for this reason, Joint B is shown in a circle in Fig. 9; in other words, the quantities listed in these two items are the distribution factors for movements of Joint B.

Item No. 3, Table 3, contains the fixed-ended forces of the two single spans under the action of given uniform load. These values have been computed by

means of diagrams and formulas given in the paper by Mr. Whitney, previously cited. If there was any known movement of the supports its influence would be calculated by Equations (5) to (20), and included with the fixed-ended forces from the load.

It follows from Item No. 3, Table 3, that there are at Joint B, an unbalanced thrust of $-$ 4.482 kips and an unbalanced moment of $-$ 25.02 kip-ft, which cause this joint to rotate through an unknown angle, α, and to move horizontally an unknown distance, Δ, so that the forces at this joint become balanced. If the rotation, α, is measured in angular units containing $\frac{1}{E}$ radians, and the displacement, Δ, in linear units of the magnitude, $\frac{100}{E}$, the numerical values of these unknowns are found from the equations of equilibrium of the joint, B: $\Sigma H_B = 0$ and $\Sigma M_B = 0$, which give:

$$h_\alpha \alpha + h_\Delta \Delta + h_{B\,(\text{fixed})} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (26)$$

and,

$$m_\alpha \alpha + m_\Delta \Delta + m_{B\,(\text{fixed})} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (27)$$

Substituting values of h and m (Joint B) from Items Nos. 1, 2, and 3, Table 3, Equations (26) and (27) become, respectively:

$$16.04\ \alpha - 6.80\ \Delta - 4.482 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (28)$$

and,

$$-634.4\ \alpha + 133.7\ \Delta - 25.02 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (29)$$

in which the coefficients before the α and Δ - terms are the distribution factors of Joint B. For Equations (28) and (29): $\alpha = -0.3541$; and, $\Delta = -1.494$.

Items Nos. 4 and 5, Table 3, give forces caused by the rotation, $0.3541\ \frac{1}{E}$, and the displacement, $1.494\ \frac{100}{E}$, of Joint B. Item No. 6 is obtained by adding, algebraically, Items Nos. 3, 4, and 5, and gives the total terminal moments and thrusts in various members of the structure under the loading considered.

In Item No. 6, values of h in Columns (3), (5), and (7), and values of m in Columns (4), (6), and (8), should add to zero. This constitutes a good check on the solution of the equations. A small discrepancy found as a result of this check has been thrown into the pier, the rigidity and distribution factors of which exceed, greatly, the corresponding values for the other members of this joint. The amounts of rotation and translation of Joint B under the action of unbalanced forces can also be found graphically. The graphical solution, however, will not be given in this paper.

Example 2.—A 4-span arch is analyzed in this more difficult example (see Fig. 10). The arch ribs and piers are alike and have the same dimensions and elastic properties as those in Example 1 (see Tables 2 and 3). The abutments and pier bases are again considered as absolutely fixed.

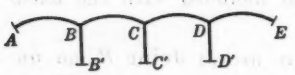

FIG. 10.—EXAMPLE 2.

Several methods of procedure can be adopted, the simplest one being, perhaps, to consider the structure as a combination of two 2-span arches, *ABC* and *CDE*, with Joint *C* fixed; and then to unlock this joint and to introduce the influence of its movements. This procedure necessitates, first, the determination of distribution factors for the movements of Joint *C*, which differ from those given by Equations (5) to (13), because the structure is extended two spans from Joint *C* instead of one.

Fig. 11 is arranged on the same basis as Table 3; the diagrams of structures in Column (2) being inserted to assist in forming mental pictures of the various steps taken, before the final solution is reached. As is evident from the diagrams, the necessary characteristics of the 4-span arch are determined after a preliminary study of 1-span and 2-span arches. The joint undergoing movement at each step is indicated by a circle around it on the diagram.

Fig. 11, Items Nos. 1 and 2, contains the already familiar distribution factors of a 2-span arch, *ABC*, with Points *A*, *B'*, and *C* fixed and Point *B* (in a circle) free to move. Item No. 3 gives rotation factors for a single arch, *BC*, when Joint *B* is fixed, and Joint *C* is permitted to move. Of course, these quantities are the same as those in Item No. 1, Columns (3), (4), and (5).

If Joint *B*, acted upon by forces in Item No. 3, is unlocked, it rotates and moves horizontally, as a center joint of a 2-span arch, so that the forces become balanced. These movements are found from formulas similar to Equations (26) and (27) by substituting the proper thrust and moment factors from Fig. 11, thus:

$$16.04\,\alpha - 6.80\,\Delta + 0.574 = 0 \qquad (30)$$

and,

$$-634.4^{\circ}\,\alpha + 133.7^{\circ}\,\Delta + 4.82 = 0 \qquad (31)$$

For clearness, Equations (30) and (31) are stated also in Fig. 11. From these equations, $\alpha = 0.0505$; and, $\Delta = 0.2035$.

The effect of these movements is expressed by the forces calculated in Items Nos. 4 and 5. When these values are added to the original forces in Arch *BC* before Joint *B* has moved (Item No. 3), the result is the final forces (Item No. 9), in various members of the double-span arch, *ABC*, when Joint *C* undergoes the specified pure rotation, $\frac{1}{E}$, and Joint *B* moves in a manner suitable to the occasion. In other words, Item No. 9 contains rotation distribution factors of the 4-span arch corresponding to movements of the center joint, *C*. The displacement factors are found similarly in Items Nos. 6, 7, 8, and 10, Fig. 11.

Items Nos. 9 and 10, Fig. 11, are completed by filling Columns C_C and C_D, and by adding, algebraically, the three C–columns to find the joint distribution factors. There is no need to fill in the various D and E–columns, since their factors are identical with those of the symmetrically opposite members, B and A. This concludes the first stage, preliminary to investigation of four cases of loading. For clearness, it may be re-emphasized that this first stage has resulted in the determination of:

(1) Terminal forces in the 2-span arch, ABC, caused by unit rotation and unit translation of Joint B (Items Nos. 1 and 2, Fig. 11).

(2) Terminal forces in the 4-span arch, $ABCDE$, caused by unit rotation and unit translation of Joint C at the center (Items Nos. 9 and 10, Fig. 11).

Item No. 11 contains the fixed-ended forces of a single arch, BC, under a uniform load (Case 1), on the right half of Span BC. Item No. 14 gives the terminal forces of the double-span arch, ABC, under the same loading, after the unbalanced forces at Joint B have been distributed, using factors in Items Nos. 1 and 2 and solving the equations given on the right side of Fig. 11 opposite Item No. 11. The only remaining step is to distribute the unbalanced forces at Joint C using the distribution factors in Items Nos. 9 and 10. The following equations of the form of Equations (26) and (27) are used:

$$16.26\,\alpha - 6.63\,\Delta + 3.675 = 0 \qquad (32)$$

and,

$$-632.00\,\alpha + 135.60\,\Delta - 21.26 = 0 \qquad (33)$$

from which the movements of the joint, C, are found to be: $\alpha = 0.1800\,\frac{1}{E}$ and $\Delta = 0.9956\,\frac{100}{E}$. The effects of movements of the joint, C, on the members in various D and E–columns are the same as on the members symmetrically located on the left side of the structure. The resultant terminal forces are entered in Item No. 17.

Case 2, in which a uniform load occupies the left half of Span BC, is investigated and the results are entered as Items Nos. 18 to 24, Fig. 11. Only two distributions of unbalanced forces are needed for its solution, since distribution factors of Joint C are already known. The equations used are all stated in Fig. 11.

It is interesting to notice that only one distribution is required for either Case 3 or Case 4, since their 2-span values can be written down directly by comparison with Cases 2 and 1. Investigation of all four cases thus requires less than twice the work on one case.

General Case of a Structure with Any Number of Spans.—Example 2 presents a good illustration of the use of the method, not only for a 4-span arch, but also for a series with any number of spans. Probably, the best procedure is to divide the structure into two parts by fixing a joint at the center (or

		Conditions	Structure Considered	A_B	B_A		$B_{B'}$		B_C		Joint B	
				m	h	m	h	m	h	m	h	m
α and Δ Factors - Joint C Moves	1	$\alpha = \frac{1}{E}$	A B C	+4.82	−0.574	−10.10	+17.19	−614.2	−0.574	−10.10	+16.04	−634.4
	2	$\Delta = \frac{100}{E}$	"	+4.78	−0.404	−4.78	−5.99	+143.3	−0.404	−4.78	−6.80	+133.7
	3	$\alpha = \frac{1}{E}$	B C						+0.574	+4.82		
	4	$\alpha = 0.0505\ \frac{1}{E}$	A B C	+0.244	−0.029	−0.510	+0.868	−31.03	−0.029	−0.510		
	5	$\Delta = 0.2035\ \frac{100}{E}$	"	+0.973	−0.082	−0.973	−1.218	+29.16	−0.082	−0.973		
	6	$\Delta = \frac{100}{E}$	B C						+0.404	+4.78		
	7	$\alpha = 0.0399\ \frac{1}{E}$	A B C	+0.192	−0.0229	−0.403	+0.686	−24.52	−0.0229	−0.403		
	8	$\Delta = 0.1533\ \frac{100}{E}$	"	+0.734	−0.0619	−0.734	−0.920	+21.98	−0.0619	−0.734		
	9	$\alpha = \frac{1}{E}$ (3)+(4)+(5)	A B C D E	+1.22	−0.111	−1.48	−0.352	−1.86	+0.463	+3.34	0.00	0.00
	10	$\Delta = \frac{100}{E}$ (6)+(7)+(8)	"	+0.93	−0.085	−1.14	−0.234	−2.50	+0.319	+3.64	0.00	0.00
Loading Case I	11	Fixed-Ended Forces	B C						−4.482	−25.02		
	12	$\alpha = -0.3542\ \frac{1}{E}$	A B C	−1.708	+0.2032	+3.577	−6.09	+217.7	+0.2032	+3.577		
	13	$\Delta = -1.496\ \frac{100}{E}$	"	−7.15	+0.604	+7.15	+8.96	−214.3	+0.604	+7.15		
	14	2-Span Forces	A B C	−8.86	+0.807	+10.73	+2.868	+3.56	−3.675	−14.29	0.00	0.00
	15	$\alpha = 0.1800\ \frac{1}{E}$	A B C D E	+0.219	−0.0200	−0.266	−0.063	−0.34	+0.083	+0.601		
	16	$\Delta = 0.9956\ \frac{100}{E}$	"	+0.926	−0.085	−1.135	−0.229	−2.49	+0.318	+3.623		
	17	4-SPAN FORCES (14)+(15)+(16)	A B C D E	−7.71	+0.702	+9.33	+2.572	+0.74	−3.274	−10.07	0.00	0.00
Loading Case II	18	Fixed-Ended Forces	B C						−4.482	+12.40		
	19	$\alpha = -0.2371\ \frac{1}{E}$	A B C	−1.143	+0.1360	+2.395	−4.074	+145.5	+0.1360	+2.395		
	20	$\Delta = -1.220\ \frac{100}{E}$	"	−5.84	+0.492	+5.84	+7.31	−174.8	+0.492	+5.84		
	21	2-SPAN FORCES (18)+(19)+(20)	A B C	−6.98	+0.628	+8.24	+3.226	−28.88	−3.854	+20.64	0.00	0.00
	22	$\alpha = 0.3232\ \frac{1}{E}$	A B C D E	+0.394	−0.0359	−0.479	−0.113	−0.615	+0.1497	+1.08		
	23	$\Delta = 1.375\ \frac{100}{E}$	"	+1.277	−0.1168	−1.566	−0.316	−3.44	+0.4386	+5.00		
	24	4-SPAN FORCES (21)+(22)+(23)	A B C D E	−5.31	+0.475	+6.20	+2.791	−32.92	−3.266	+26.72	0.00	0.00
Loading Case III	25	2-Span Forces	A B C	−18.04	+3.854	−20.64	−3.226	+28.88	−0.628	−8.24	0.00	0.00
	26	$\alpha = 0.06612\ \frac{1}{E}$	A B C D E	+0.081	−0.0073	−0.098	−0.0233	−0.123	+0.0306	+0.221		
	27	$\Delta = 0.2568\ \frac{100}{E}$	"	+0.239	−0.0218	−0.293	−0.0601	−0.642	+0.082	+0.935		
	28	4-SPAN FORCES (25)+(26)+(27)	A B C D E	−17.72	+3.825	−21.03	−3.310	+28.11	−0.515	−7.08	0.00	0.00
Loading Case IV	29	2-Span Forces	A B C	+21.26	+3.675	+14.29	−2.868	−3.56	−0.807	−10.73	0.00	0.00
	30	$\alpha = 0.0847\ \frac{1}{E}$	A B C D E	+0.103	−0.0094	−0.125	−0.0298	−0.157	+0.0392	+0.282		
	31	$\Delta = 0.3292\ \frac{100}{E}$	"	+0.306	−0.028	−0.375	−0.0771	−0.824	+0.105	+1.198		
	32	4-SPAN FORCES (29)+(30)+(31)	A B C D E	+21.67	+3.638	+13.79	−2.975	−4.54	−0.663	−9.25	0.00	0.00
									ALTERNATIVE			
m and h Factors Joint B Moves	33	$\alpha = 0.313\ \frac{1}{E}$	A B C	+1.510	−0.1796	−3.161	+5.38	−192.2	−0.1796	−3.161		
	34	$\Delta = 0.739\ \frac{100}{E}$	"	+3.530	−0.2984	−3.530	−4.427	+105.95	−0.2984	−3.530		
	35	$\alpha = 0.0616\ \frac{1}{E}$	"	+0.297	−0.0353	−0.6222	+1.0595	−37.82	−0.0353	−0.6222		
	36	$\Delta = 0.2921\ \frac{100}{E}$	"	+1.396	−0.1180	−1.3965	−1.750	+41.9	−0.1180	−1.396		
	37	$m = 100$ (33)+(34)	"	+5.04	−0.478	−6.69	+0.956	−86.62	−0.478	−6.69	0.00	−100.00
	38	$h = 1$ (35)+(36)	"	+1.693	−0.1533	−2.019	−0.6934	+4.038	−0.1533	−2.019	−1.000	0.00
m and h Factors Joint C Moves	39	$\alpha = 0.3339\ \frac{1}{E}$	A B C D E	+0.407	−0.0371	−0.494	−0.1175	−0.621	+0.1546	+1.115		
	40	$\Delta = 0.819\ \frac{100}{E}$	"	+0.761	−0.0696	−0.933	−0.1916	−2.048	+0.2613	+2.981		
	41	$\alpha = 0.0683\ \frac{1}{E}$	"	+0.083	−0.0076	−0.101	−0.0240	−0.127	+0.0316	+0.228		
	42	$\Delta = 0.3182\ \frac{100}{E}$	"	+0.296	−0.0271	−0.363	−0.0745	−0.796	+0.1015	+1.159		
	43	$m = 100$ (39)+(40)	"	+1.17	−0.107	−1.43	−0.309	−2.67	+0.416	+4.10	0.00	0.00
	44	$h = 1$ (41)+(42)	"	+0.379	−0.0347	−0.464	−0.0984	−0.923	+0.1331	+1.387	0.00	0.00
Loading Case I'	45	Fixed-Ended Forces	H C						−4.482	−25.02		
	46	$m = -25.02$	A H C	−1.260	+0.1196	+1.674	−0.2392	+21.67	+0.1196	+1.674		
	47	$h = -4.482$	"	−7.60	+0.6878	+9.053	+3.107	−18.09	+0.6878	+9.053		
	48	2-SPAN FORCES (45)+(46)+(47)	A H C	−8.86	+0.808	+10.73	+2.866	−3.56	−3.674	−14.29	0.00	0.00
	49	$m = -21.26$	A B C D E	−0.249	+0.0228	+0.304	+0.0657	+0.568	−0.0885	−0.872		
	50	$h = +3.674$	"	+1.394	−0.1275	−1.703	−0.3615	−3.392	+0.4895	+5.100		
	51	4-SPAN FORCES (48)+(49)+(50)	A B C D E	−7.71	+0.703	+9.33	+2.570	+0.73	−3.273	−10.06	0.00	0.00

FIG. 11.—END MOMENTS AND THRUSTS OF A FOUR-

C_B		$C_{C'}$		C_D		Joint C		D_C		$D_{D'}$		D_E		Joint D		E_D
h	m	h	m	h	m	h	m	h	m	h	m	h	m	h	m	m
+0.574	+4.82															
+0.404	+4.78															
−0.574	−10.10							$16.04\alpha - 6.80\Delta + 0.574 = 0$								
+0.029	+0.244							$-634.4\alpha + 133.7\Delta + 4.82 = 0$								
+0.082	+0.973															
−0.404	−4.78							$16.04\alpha - 6.80\Delta + 0.404 = 0$								
+0.0229	+0.192							$-634.4\alpha + 133.7\Delta + 4.78 = 0$								
+0.0619	+0.734															
−0.463	−8.88	+17.19	−614.2	−0.463	−8.88	+16.26	−632.0									
−0.319	−3.85	−5.99	+143.3	−0.319	−3.85	−6.63	+135.6									
+4.482	−12.40							$16.04\alpha - 6.80\Delta - 4.482 = 0$								
−0.2032	−1.708							$-634.4\alpha + 133.7\Delta - 25.02 = 0$								
−0.604	−7.15															
+3.675	−21.26							$16.26\alpha - 6.63\Delta + 3.675 = 0$								
−0.083	−1.60	+3.094	−110.6	−0.083	−1.60			$-632.0\alpha + 135.6\Delta - 21.26 = 0$								
−0.318	−3.83	−5.96	+142.6	−0.318	−3.83			+0.083	+0.601	−0.063	−0.34	−0.020	−0.266			+0.219
+3.274	−26.69	−2.873	+32.12	−0.401	−5.43	0.00	0.00	+0.318	+3.623	−0.229	−2.49	−0.085	−1.135			+0.926
								+0.401	+4.22	−0.296	−2.82	−0.105	−1.40	0.00	0.00	+1.15
+4.482	+25.02															
−0.1360	−1.143							$16.04\alpha - 6.80\Delta - 4.482 = 0$								
−0.492	−5.84							$-634.4\alpha + 133.7\Delta + 12.40 = 0$								
+3.854	+18.04															
−0.1497	−2.87	+5.56	−198.4	−0.1497	−2.87			$16.26\alpha - 6.63\Delta + 3.854 = 0$								
−0.4386	−5.29	−8.23	+196.9	−0.4386	−5.29			$-632.0\alpha + 135.6\Delta + 18.04 = 0$								
+3.266	+9.88	−2.678	−1.72	−0.588	−8.16	0.00	0.00	+0.1497	+1.08	−0.113	−0.615	−0.0359	−0.479			+0.394
								+0.4386	+5.00	−0.316	−3.44	−0.1168	−1.566			+1.277
								+0.588	+6.08	−0.435	−4.04	−0.153	−2.04	0.00	0.00	+1.67
+0.628	+6.98															
−0.0306	−0.587	+1.137	−40.6	−0.0306	−0.587			$16.26\alpha - 6.63\Delta + 0.628 = 0$								
−0.082	−0.989	−1.539	+36.8	−0.082	−0.989			$-632.0\alpha + 135.6\Delta + 6.98 = 0$								
+0.515	+5.40	−0.402	−3.82	−0.113	−1.58			+0.0306	+0.221	−0.0233	−0.123	−0.0073	−0.098			+0.081
								+0.082	+0.935	−0.0601	−0.642	−0.0218	−0.293			+0.239
								+0.113	+1.16	−0.084	−0.77	−0.029	−0.39			+0.32
+0.807	+8.86															
−0.0392	−0.752	+1.455	−52.0	−0.0392	−0.752			$16.26\alpha - 6.63\Delta + 0.807 = 0$								
−0.105	−1.267	−1.97	+47.15	−0.105	−1.267			$-632.0\alpha + 135.6\Delta + 8.86 = 0$								
+0.663	+6.84	−0.519	−4.82	−0.144	−2.02	0.00	0.00	+0.0392	+0.282	−0.0298	−0.157	−0.0094	−0.125			+0.103
								+0.105	+1.198	−0.0771	−0.824	−0.028	−0.375			+0.306
								+0.144	+1.48	−0.107	−0.98	−0.037	−0.50	0.00	0.00	+0.41
		METHOD														
+0.1796	+1.510							$16.04\alpha - 6.80\Delta = 0$								
+0.2984	+3.530							$-634.4\alpha + 133.7\Delta + 100 = 0$								
+0.0353	+0.297							$16.04\alpha - 6.80\Delta + 1 = 0$								
+0.1180	+1.396							$-634.4\alpha + 133.7\Delta = 0$								
+0.478	+5.04															
+0.1533	+1.693															
−0.1546	−2.964	+5.74	−205.0	−0.1546	−2.964			$16.26\alpha - 6.63\Delta = 0$								
−0.2613	−3.152	−4.908	+117.4	−0.2613	−3.152			$-632.0\alpha + 135.6\Delta + 100 = 0$								
−0.0316	−0.606	+1.173	−41.92	−0.0316	−0.606			$16.26\alpha - 6.63\Delta + 1 = 0$								
−0.1015	−1.225	−1.905	+45.6	−0.1015	−1.225			$-632.0\alpha + 135.6\Delta = 0$								
−0.416	−6.12	+0.832	−87.76	−0.416	−6.12	0.00	−100.00									
−0.1331	−1.831	−0.7338	+3.662	−0.1331	−1.831	−1.000	0.00									
+4.482	−12.40															
−0.1196	−1.260															
−0.6878	−7.60															
+3.674	−21.26															
+0.0885	+1.301	−0.1768	+18.65	+0.0885	+1.301			−0.0885	−0.872	+0.0657	+0.568	+0.0228	+0.304			−0.249
−0.4895	−6.740	−2.695	+13.45	−0.4895	−6.740			+0.4895	+5.100	−0.3615	−3.392	−0.1275	−1.703			+1.394
+3.273	−26.70	−2.872	+32.14	−0.401	−5.44	0.00	0.00	+0.401	+4.23	−0.296	−2.83	−0.105	−1.40	0.00	0.00	+1.15

SPAN ARCH UNDER UNIFORM LOAD, $w = 0.1$ KIP PER FOOT.

near the center for an odd number of spans), to analyze each part separately, and then, having determined distribution factors for movement of the temporarily fixed joint, to distribute the unbalanced forces at that joint.

After the preliminary work of calculating distribution factors, the necessary distributions for the load require relatively little time. In designing a multiple arch there are several cases of loading under which the structure should be investigated, and when using this method, the more cases are analyzed the less work on the average is involved in each, as has been already demonstrated in connection with Example 2.

The method can also be applied to the construction of influence lines. An influence line for moment or thrust at any point can be found by the use of Maxwell's reciprocal theorem, as a deflection curve of a suitably deformed structure after the unbalanced terminal forces caused by the deformation have been distributed in the usual manner. This procedure is cumbersome, because it requires the translation of moments into deflections.

However, a complete set of influence lines of all terminal forces (and this is what is generally required in the final analysis of the structure, if influence lines are used at all) can be found comparatively easily, following closely the procedure of Example 2, by placing a unit load, successively, at different points of the arch. In Example 2 this method requires only twenty-seven distributions to construct influence lines for all ten terminal forces, having ordinates at the tenth-points of the spans. This amount of work is not unreasonable in view of the great number of force functions.

Alternative Method.—In cases requiring a large number of distributions, a change in the procedure is recommended which does away with the equations determining the necessary movements of the joints. This modified procedure is presented, as applied to Case 1 under the sub-title, "Alternative Method," in Fig. 11. The idea is to replace the rotation and displacement factors with, what may be termed, m-factors and h-factors. There are again four of these factors at each end of each member: (1) The m-moment factor, m_m; (2) the m-thrust factor, h_m; (3) the h-moment factor, m_h; and (4) the h-thrust factor, h_h.

The first two are, respectively, the moment and the thrust that occur at the end of any member (as, for example, A_B), when the joint under consideration (say, B) is acted upon by a moment of 1, or 100, created by some external agency. As a result of this moment, Joint B undergoes both rotation, α, and horizontal translation, Δ, the values of which can be found easily, and from these movements the moments and thrusts at the ends of all members (the m-factors) can be computed. Similarly, the h-factors are the moments and thrusts at different terminals when the joint in question is acted upon by a thrust, $h = 1$.

The m-factors and h-factors for a 2-span arch, ABC, with movement at Joint B are determined in Items Nos. 33 to 38, Fig. 11, by the use of rotation and displacement factors in Items Nos. 1 and 2. Equations necessary for finding the movements of Joint B, when it is acted upon by a moment, 100,

are as follows:

$$16.04\ \alpha - 6.80\ \Delta = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (34)$$

and,

$$-\ 634.40\ \alpha + 133.70\ \Delta + 100.00 = 0 \ldots\ldots\ldots\ldots\ldots (35)$$

whence $\alpha = 0.313$ and $\Delta = 0.739$.

Forces caused by these movements are added in Item No. 37, Fig. 11, which thus presents the m-factors of the 2-span arch, ABC. The h-factors, found in a similar manner, are recorded in Item No. 38. Items Nos. 43 and 44, Fig. 11, contain the m-factors and h-factors for a 4-span arch with motion at the center joint, C; they are found by following a similar procedure.

These four additional distributions involved in determining m-factors and h-factors eliminate the equations when investigating the influence of unbalanced fixed-ended forces. Thus, in the 2-span arch, ABC, under the loading of Case 1 (Items Nos. 45 to 48, Fig. 11), the influence of the fixed-ended thrust, — 4.482 kips, at Joint B, is found at Item No. 47 as a product of this quantity and the corresponding h-factors, without resort to equations. The same is true with regard to the unbalanced moment, — 25.02 kip-ft.

Conclusions

The following conclusions are advanced as to the value and limitations of the method presented in this paper:

(1) With the exception of evaluation of distribution factors for single members (which is not considered a requisite part of this analysis), the method does not involve mathematics higher than elementary algebra.

(2) The idea behind the method is simple, and the various steps in the solution are quite easy to visualize, especially when accompanied by small supplementary diagrams.

(3) The nature of the computations, involving multiplication of a string of numbers by a single factor, makes the method particularly suitable for slide-rule work. (All computations in Table 3 and Fig. 11 have been made with an ordinary 10-in. slide-rule.)

(4) The orderly tabulation diminishes a possibility of errors.

(5) The algebraic sum of the terminal forces for three members meeting at a joint must equal zero. This provides an effective check on the results, the only exception being the moments at the abutments. In fact, there is little chance for mistake, after the fixed-ended forces and the single-span distribution factors have been determined correctly.

(6) Perhaps the part of the method most likely to cause confusion is the convention pertaining to the meaning and signs of distribution factors. The signs of some of the single-member distribution factors, when observed casually (see Items Nos. 1 and 2, Fig. 11), may seem quite odd. However, once the convention is mastered, and one visualizes the movement of the end of the member in relation to its neutral point, a good reason becomes apparent for each sign.

(7) The examples solved have dealt with a series of equal symmetrical arches, but there is no change when individual arches are different but sym-

metrical. The method still applies to asymmetrical arches, although Expressions (1) to (14), for distribution factors, are no longer valid.

(8) Vertical settlement of the joints due to pier-shortening has been disregarded; it is not important, but can be taken care of independently afterward, if desired. With the exception of this theoretical limitation, the method is "exact" in a sense that it is not based on assumptions other than the ordinary ones of the theory of elasticity.

(9) While, perhaps, this method may prove more laborious than some others when investigating one load condition, the writer believes that time will be saved when applying it to several load conditions, or to the construction of influence lines. It may be mentioned in this connection that a casual inspection of Fig. 11 is likely to convey an exaggerated idea of the labor involved in the solution. The fact is that many of the values are merely copied from one column into another, and any one acquainted with the procedure will find simplifications and "short-cuts" that will lessen the work.

In conclusion, the writer wishes to express the hope that the method will find its way into designing offices.

Acknowledgment

The writer wishes to acknowledge his indebtedness to A. H. Finlay, Assoc. M. Am. Soc. C. E., for reading the manuscript and for most valuable criticism and suggestions.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

PAPERS

RATIONAL DESIGN OF STEEL COLUMNS

By D. H. Young,[1] Jun. Am. Soc. C. E.

Synopsis

A basis for the design of steel columns is offered in this paper. The effect of general imperfections is represented by an assumed form of initial curvature of the axis of the column, or by an eccentricity in the application of the load; the loading first producing yielding in the most stressed fibers, due to the assumed curvature or eccentricity, is then used as the criterion for the selection of the working load.

The proposed method is applicable to pin-ended columns under several conditions of loading, design formulas being developed for each case. In addition, an approximate solution is made for the case of columns in rigid frame construction.

The question of shear in built-up steel columns is treated on the same basis and for the same conditions of loading, formulas being developed for the design of lacing or batten-bars in such columns.

The paper represents, in somewhat condensed form, work done by the writer in preparing his thesis[2] entitled "Rational Design of Steel Columns."

Introduction

Since the time of Euler, the question of the resistance and stability of compression members has been controversial. The inherent difficulty with the column is that slight imperfections have a pronounced effect upon its behavior under load. The chief factors that affect the behavior of the column may be listed, as follows: (1) Imperfect elasticity of the material; (2) initial crookedness of the axis; (3) accidental eccentricity in the application of load; and (4) uncertainty of conditions at the ends of the column. Theories that do not take into account the extent of the effect of such imperfections are of little practical value to the designer.

Note.—Presented at the Joint Meeting of the Structural Division, Am. Soc. C. E., and Applied Mechanics Division, Am. Soc. Mech. Engrs., Chicago, Ill., June 29, 1933. Discussion on this paper will be closed in March, 1935, *Proceedings*.

[1] Instructor in Eng. Mechanics, Univ. of Michigan, Ann Arbor, Mich.

[2] Submitted to the Univ. of Michigan in 1934, in partial fulfillment of the requirements for the degree of Doctor of Science in Civil Engineering.

As a result of such influences as Factors (1) to (4), it is present-day practice to design columns by empirical formulas. Because such formulas are backed by considerable experience, and because they generally use a liberal factor of safety, they have proved satisfactory, in most cases. The designer, however, has at his disposal such a variety of formulas that widely different results can be obtained for the same column. Curves are available[3] showing the relation between the allowable average compressive stress and the slenderness ratio, as prescribed for steel columns by various building codes and specifications in the United States. Taken altogether, such curves represent a band across the range of slenderness ratios fully 10 000 lb per sq in. wide. Of course, in all justification, it must be admitted that they represent a wide range of conditions, and each of them carries with it a number of restrictions upon its use, so that, finally, there is not the utter state of confusion that might be apparent at first glance. However, such a collection of curves is far from representing any unified agreement regarding the strength of steel compression members.

Granting the sufficiency of such empirical formulas for ordinary cases, it seems desirable to have a more general theoretical basis for design, which will take into account as far as possible all necessary factors. A rational formula has the advantage of being consistent over a much wider range of conditions than the empirical formula. Furthermore, it may show how Factors (1) to (4) affect the strength of the column, and thus the designer can make some saving in material by a careful control of workmanship or material; again, experiments conducted in the light of some kind of a rational theory, even if it is far from perfect, can yield far more valuable results than experiments conducted more or less blindly.

Notation

The symbols used throughout the paper are given in the Appendix. An effort has been made to conform as nearly as practicable with the "Symbols for Mechanics, Structural Engineering, and Testing Materials" advanced by the American Standards Association.[4]

PART I.—COLUMNS OF SOLID CROSS-SECTION

1.—Basis for Design

Nature of the Material.—A typical stress-strain diagram for ordinary mild steel is shown in Fig. 1. The curve has three significant points: (1) The proportional limit; (2) the yield point; and (3) the ultimate strength. One of these three points is usually taken as a basis when considering allowable working stresses.

At one time working stresses used in design were based on the ultimate strength of the material. This was illogical because, at the same time, stresses were computed on the assumption that Hooke's law applied. On such a basis the designer had no such factor of safety against failure as he assumed.

[3] For example, see "Steel Construction," Manual of the Am. Inst. of Steel Construction, First Edition, 1933, p. 176.

[4] A S A—Z10*a*—1932.

At present, the yield point of the steel is quite generally recognized as the limit of usefulness. The deformation which takes place during yielding is generally from ten to fifteen times the elastic deformation at the proportional limit. Such yielding, while not properly representing complete failure, results in structural damage which cannot be allowed in ordinary structures.

To analyze a structure, the ordinary equations of elasticity based upon Hooke's law can be used, strictly speaking, only for stresses within the proportional limit. For stresses beyond this the true shape of the stress-strain diagram (Fig. 1) must be taken into account, and the problem frequently

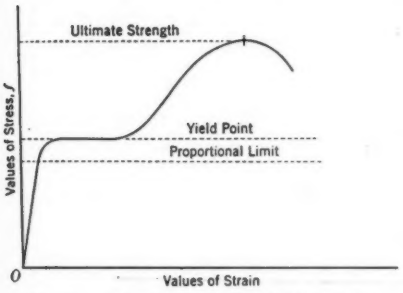

FIG. 1.—TYPICAL STRESS-STRAIN DIAGRAM FOR MILD STEEL.

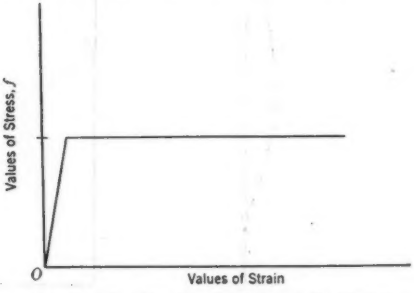

FIG. 2.—IDEAL STRESS-STRAIN DIAGRAM.

becomes very complex. In general, the effect of a slight deviation from a straight line between the proportional limit and the yield point will be small and can be safely neglected.

Assuming that the material is perfectly elastic up to the yield point, and interpreting yielding as failure, amounts to assuming a stress-strain diagram such as that shown in Fig. 2. It is, therefore, expedient and justifiable, in every way, to work with such an ideal stress-strain curve, and this is the usual practice. Furthermore, it is present-day practice to select the allowable loads for beams, subjected to bending, on the basis of the load that first produces a stress in the extreme fibers equal to the yield point, notwithstanding the fact that the beam will not collapse completely until all the inner fibers have begun to yield or until the extreme fibers have actually ruptured. It is well known that a beam of I-section is much nearer complete collapse when the extreme fibers first begin to yield than a beam of circular or rectangular cross-section. It is not usual practice in structural design, however, to take any account of this reserve strength against complete collapse in the case of the circular or rectangular cross-section; but rather to take the beginning of yielding in the extreme fibers as the criterion for design. In other words, this kind of structural damage is interpreted as failure.

Thus, all difficulty with Factor (1) (imperfect elasticity of the material) is eliminated. It will be logical to approach the problem of column design in the same way. To establish the logic of such a treatment, it will be convenient to consider qualitatively the general behavior of columns under load before going into more detail.

A Perfectly Straight, Axially Loaded, Pin-Ended Steel Column.—If the perfectly straight, pin-ended, steel column shown solid in Fig. 3(*a*), is slender, so that the Euler load may be reached before the fiber stresses become equal to the proportional limit, the load-deflection diagram is similar to Curve *OABC* in Fig. 3(*b*). At first, as the load increases, there is no lateral deflec-

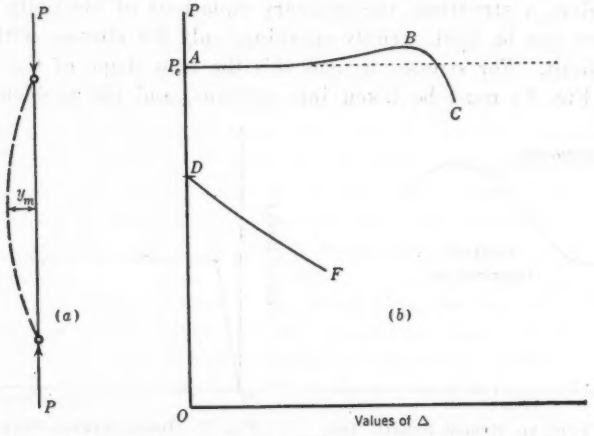

FIG. 3.—BEHAVIOR OF AN INITIALLY STRAIGHT, AXIALLY LOADED COLUMN.

tion. At the Euler load, P_e, the column becomes elastically unstable and may have any small deflection. A very slight increase in load above the Euler value produces a large lateral deflection. (For an increase of load 1% above the Euler value the lateral deflection at the center becomes in the order of 9% of the length of the bar.) As the load increases there finally comes a point beyond which the load necessary to maintain further deflection falls off rapidly. Before this occurs, however, the stresses in the extreme fibers will have passed the yield point. Point *B*, corresponding to the maximum load which the column can carry, represents the buckling load and is slightly greater than the Euler load.

If the column is short enough so that the average compressive stress becomes equal to the yield-point stress before the Euler load is reached, the load-deflection diagram will be similar to Curve *ODF* in Fig. 3(*b*). The column remains straight for loads up to the maximum load at Point *D*, which corresponds approximately to the yield point, and then buckles suddenly, the load necessary to maintain any deflection falling off rapidly as the deflection is increased. The theory of such inelastic buckling was first developed by F. Engesser[5] in 1891.

Initially Curved or Eccentrically Loaded, Pin-Ended Steel Columns.—When a slender column is initially curved, or straight and eccentrically loaded, as represented in Fig. 4(*a*) and Fig. 4(*b*), the load deflection diagram will be similar to Curve *OABC* in Fig. 4(*c*). For any value of the load there

[5] "Die Knickfestigkeit gerader Stäbe," von F. Engesser, *Zentralblatt der Bauverwaltung*, Berlin, 1891.

is a definite lateral deflection. This deflection increases very slowly until the load approaches the Euler value, after which it begins to increase rapidly to some point, beyond which the load necessary to maintain further deflection falls off. Point B, corresponding to the maximum load which the column can carry, represents the buckling load. Before this load is reached the maximum fiber stresses will have passed the yield point.

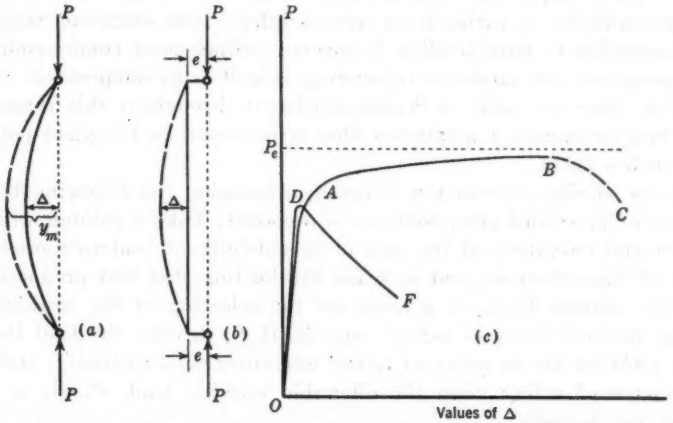

FIG. 4. — BEHAVIOR OF AN INITIALLY CURVED OR ECCENTRICALLY LOADED COLUMN.

If the column is short, the load-deflection diagram will be similar to Curve ODF in Fig. 4(c). Deflection increases slowly to Point D, and then the column buckles suddenly, the load necessary to maintain further deflection falling off rapidly as the deflection is increased. The theory of such inelastic buckling for initially curved or eccentrically loaded steel columns was first developed by Theodor von Kármán, M. Am. Soc. C. E., in 1910.[6] Professor von Kármán also obtained such curves as those in Fig. 4(c), experimentally, and succeeded in getting remarkable agreement with the theory.

The determination of the curves, as shown in Fig. 4(c), must take account of the slenderness ratio of the column, the shape of the cross-section, the true characteristics of the stress-strain diagram (Fig. 1), and the manner of loading. The labor involved in making these calculations is great, and also a definite stress-strain diagram must be assumed. Since the variation in the yield point of steel for various specimens is probably as much as ± 10%, this limitation would not seem to warrant such refinement in calculating buckling loads, for purposes of design, as has been outlined.

Furthermore, due to the sudden nature of inelastic buckling, it is obvious that the fiber stresses have gone past the yield point before complete buckling occurs. It is doubtful, therefore, whether the buckling load should be used as a criterion for the design of columns any more than the loads that produce

[6] "Untersuchungen über Knickfestigkeit," *Forschungsarbeiten*, Nr. 81, 1910; see, also, "Strength of Steel Columns," by H. M. Westergaard, M. Am. Soc. C. E., and William R. Osgood, Assoc. M. Am. Soc. C. E., *Transactions*, A. S. M E., Vol. 49–50 (Paper APM 50–9), 1927–28.

complete collapse of a beam should be used as the criterion for the selection of design loads. It would seem more consistent to design columns on a basis of the loading that first produces extreme fiber stresses equal to the yield point, as is done in the case of beams.

Finally, the introduction of initial curvature or eccentricity of load makes the column problem always one of combined bending and direct stress. Consider, for example, the eccentrically loaded column shown in Fig. 4(*b*). As the eccentricity, e, varies from zero to infinity, the complete range from pure compression to pure bending is covered. Since most compression members in structures are subjected to bending as well as to compression, it seems desirable to have one basis of design consistent throughout this range. The loading first producing a maximum fiber stress equal to the yield point will furnish such a basis.

Basis for Design.—From the foregoing discussion the following basis for the design of pin-ended steel columns is proposed: Take a column with some definite initial curvature of the axis or eccentricity of load to represent the effect of all imperfections, and then use the loading that first produces yielding in the extreme fibers as a basis for the selection of the working load, using any desired factor of safety; that is, if P_y denotes the load that first produces yielding due to assumed initial curvature or eccentricity, and n, the desired factor of safety, then the allowable working load, P_w, is to be obtained by the formula:

$$P_w = \frac{P_y}{n} \qquad \text{.....(1)}$$

2.—Imperfections Represented by Initial Curvature

The purpose of studying the behavior of an initially curved bar under compressive loads will be to learn the effect of possible accidental initial

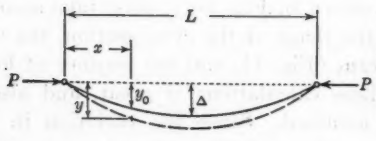

Fig. 5.—Initially Curved Pin-Ended Column.

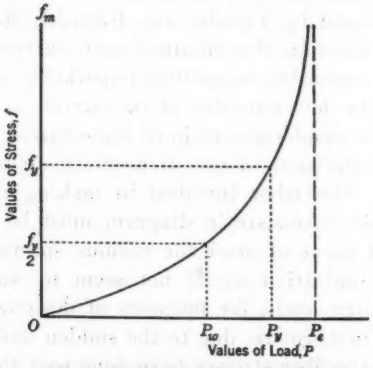

Fig. 6.—Non-Linear Relationship Between Axial Load and Maximum Fiber Stress.

curvature on the strength of the column. Such accidental initial curvature of the bar may be of almost any shape. For a given initial deflection the most serious condition for the bar (as far as bending stresses are concerned)

will be a curve, such that all points on the axis of the bar are displaced on the same side of a straight line through the ends. Various types of curvature satisfying this condition are the arc of a circle, a parabola, a half sine curve, etc. Since actual accidental initial curvature is as likely to be approximated by one of these curves as another, it seems logical to assume the form that can be most simply handled mathematically. For this reason the half wave of a sine curve will be chosen.

Consider a bar initially curved as shown in Fig. 5. The maximum deviation from a straight line at the center of the bar is Δ, and the initial deflection at any other point distant, x, from the left end, will be given by the expression:

$$y_o = \Delta \sin \frac{\pi x}{L} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

When such an initially curved bar is acted upon by compressive end loads, P, as shown, the bar assumes a definite shape for each value of the load.

Choosing the original of co-ordinates at the left end of the bar, the general relation between curvature and bending moment at any section will be,

$$E I \left(\frac{d^2 y}{dx^2} - \frac{d^2 y_o}{dx^2} \right) = - P y \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Equation (3), of course, holds only for a fairly flat deflection curve; also it is based on the assumption that Hooke's law applies. The possibility of buckling perpendicular to the plane of bending is not considered. Substituting the value of $\frac{d^2 y_o}{dx^2}$ from Equation (2) and letting the quantity, $\frac{P}{EI} = q^2$, gives:

$$\frac{d^2 y}{dx^2} + q^2 y = - \frac{\pi^2}{L^2} \Delta \sin \frac{\pi x}{L} \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Equation (4) represents the differential equation of the elastic line of the bar and its solution is,

$$y = C_1 \sin q x + C_2 \cos q x + \frac{\Delta \pi^2}{\pi^2 - q^2 L^2} \sin \frac{\pi x}{L} \quad \ldots\ldots\ldots (5)$$

in which, C_1 and C_2 are constants to be evaluated from the end conditions of the bar. These constants will be: $C_2 = 0$; and $C_1 \sin q L = 0$. From the latter of these two conditions either $C_1 = 0$, or $\sin q L = 0$. If $\sin q L = 0$ then either $q L = 0$, or $q L = i \pi$, in which, i is any integer. From these two conditions either $P = 0$, or $P = \frac{i^2 \pi^2 E I}{L^2}$. The condition, $P = 0$, obviously holds no significance. For the second condition, P is seen to represent the Euler load for the bar, in which case the deflection becomes infinite. It follows, then that for finite deflections, $C_1 = 0$ must be taken. Equation (5) then reduces to,

$$y = \frac{\Delta \pi^2}{\pi^2 - q^2 L^2} \sin \frac{\pi x}{L} \quad \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

Equation (6) defines the axis of the bar for any value of the load between $P = 0$ and $P = P_e$. The deflection will be a maximum at the center of the bar. Making $x = \frac{L}{2}$, in Equation (6), gives,

$$y_m = \frac{\Delta \pi^2}{\pi^2 - q^2 L^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

Substituting $q^2 = \frac{P}{EI}$, and $\frac{\pi^2 EI}{L^2} = P_e$, Equation (7) may be written:

$$y_m = \Delta \left(\frac{1}{1 - \frac{P}{P_e}} \right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

The maximum bending moment, at the center of the bar, will be $P \times y_m$, or,

$$M_m = \Delta \left(\frac{P}{1 - \frac{P}{P_e}} \right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

The maximum fiber stress, in the outside fiber, on the concave side of the bar will be,

$$f_m = \frac{P}{A} + \frac{M_m}{S} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10)$$

in which, S is the section modulus of the cross-section. Substituting the value of M_m into Equation (10) gives,

$$f_m = s \left[1 + \frac{\Delta}{k} \left(\frac{1}{1 - \frac{s}{s_e}} \right) \right] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11)$$

in which, $k = \frac{S}{A}$ is the core radius of the cross-section; $s = \frac{P}{A}$ is the load per unit area or average compressive stress; and $s_e = \frac{P_e}{A} = \frac{\pi^2 E}{\left(\frac{l}{r}\right)^2}$. The use of the core radius, k, makes possible a simple interpretation of Equation (11). If a short block is compressed by a load, P, applied with an eccentricity, $e = \Delta$, the maximum compressive stress from Equation (10) becomes, $f_m = \frac{P}{A}\left(1 + \frac{\Delta}{k}\right)$. The term, $\frac{\Delta}{k}$, then represents the ratio of the bending stress to the direct compressive stress. The term, $\left(\frac{1}{1 - \frac{s}{s_e}} \right)$, in Equation (11), may now be considered as a magnification factor which shows the effect, on

the bending stress, of taking into account the deflections of the bar. The ratio, $\frac{\Delta}{k}$, will be called the eccentricity ratio. Equation (11) gives the maximum fiber stress for any given value of the load, P, and any amount of initial curvature as represented by Δ.

Making $f_m = f_y$ and, at the same time, denoting the corresponding average compressive stress by s_y, Equation (11) may be written:

$$s_y = \frac{f_y}{1 + \frac{\Delta}{k}\left(\frac{1}{1 - \frac{s_y}{s_e}}\right)} \qquad (12)$$

which gives the load per unit area first producing a maximum fiber stress equal to the yield point.

If values of f_m computed from Equation (11) are plotted against the load, P, a curve such as that shown in Fig. 6 is obtained. It is seen from this curve that no proportionality exists between load and maximum fiber stress. This means that merely to assume a safe working stress, say, one-half the yield point (Fig. 6), and to use this in Equation (11) to compute the safe working load, is not permissible. Any desired factor of safety, however, may be incorporated directly into Equation (11) as follows: Assume that P_w is the allowable load and n is a factor of safety, such that the maximum stress becomes equal to the yield point when the load becomes equal to $n\ P_w$. If, in Equation (11), the load per unit area, s, is replaced by $n\ s_w$, and, at the same time, f_y is substituted for f_m, Equation (11) becomes:

$$s_w = \frac{f_y}{n + \frac{n\,\Delta}{k}\left(\frac{1}{1 - \frac{n\,s_w}{s_e}}\right)} \qquad (13)$$

The value of s_w computed from Equation (13) will always be such that n times this value will produce a maximum fiber stress equal to the yield point, and, hence, the desired factor of safety is realized. The appearance of n in the magnification factor then takes care of the lack of proportionality between load and fiber stress. This manner of incorporating a factor of safety, n, in formulas similar to Equation (13), was proposed by K. S. Zavriev,[7] who also calculated tables of allowable average stresses for various kinds of combined bending and compression of bars.

Curves may also be plotted from Equation (12) for any given value of f_y and various values of the eccentricity ratio, $\frac{\Delta}{k}$. These curves will show the average compressive stress, s_y, as a function of the slenderness-ratio, $\frac{L}{r}$, at which yielding will first begin. Any desired factor of safety, n, may also be obtained by simply dividing the value of s_y, read from the curve, by the de-

[7] *Bulletin*, Soc. of the Engrs. of Technology, St. Petersburg, 1913.

sired factor of safety. Such a set of curves for a steel having a yield point of 36 000 lb per sq in., and values of $\frac{\Delta}{k}$ ranging from 0 to 1.0, is shown in Fig. 7.

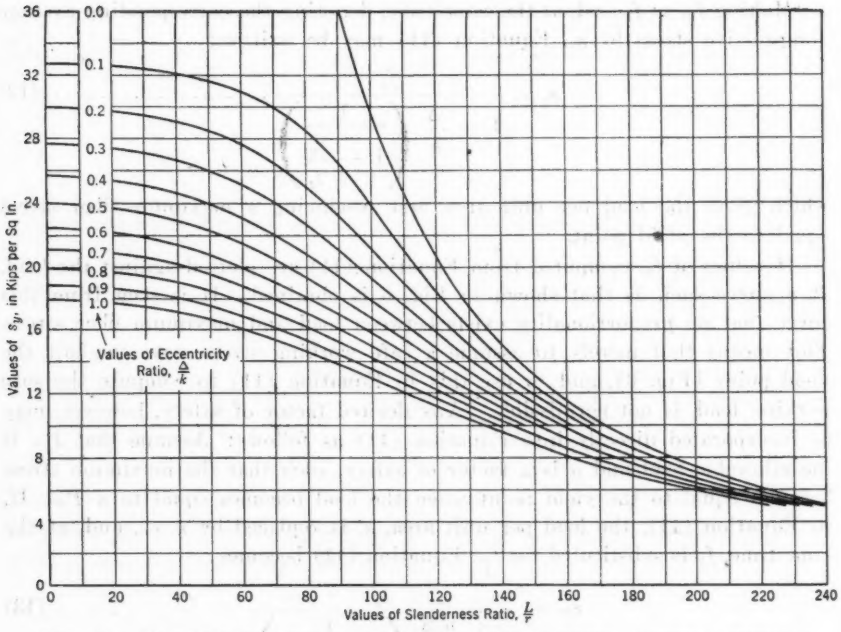

FIG. 7. — YIELD-POINT LOAD FOR VARIOUS ECCENTRICITY RATIOS; INITIALLY CURVED COLUMN.

It will be interesting to compare the values of the average compressive stress that first produce yielding with those[6] that cause buckling (complete failure) of the column, as discussed in Section 1. In Fig. 8, this comparison is made for the same steel, having a yield point of 36 000 lb per sq in., a proportional limit of 30 000 lb per sq in., and values of the eccentricity ratio ranging from 0.0 to 0.9. The curves for buckling apply only for columns of rectangular cross-section. For columns of I-section, the difference between corresponding curves will be less. The solid line curves show the value of average compressive stress at which yielding first begins, whereas the broken lines represent complete buckling. It will be seen from a study of these curves that for large values of $\frac{\Delta}{k}$ (more bending in proportion to direct stress), the buckling load is considerably greater for medium long columns than the yield-point load. As $\frac{\Delta}{k}$ decreases, the difference between corresponding curves

becomes less. For values of $\frac{\Delta}{k}$ less than about 0.1, Equation (12) gives values of the load greater than the buckling load. This discrepancy on the side of danger, for nearly perfect columns, is of no consequence, since in

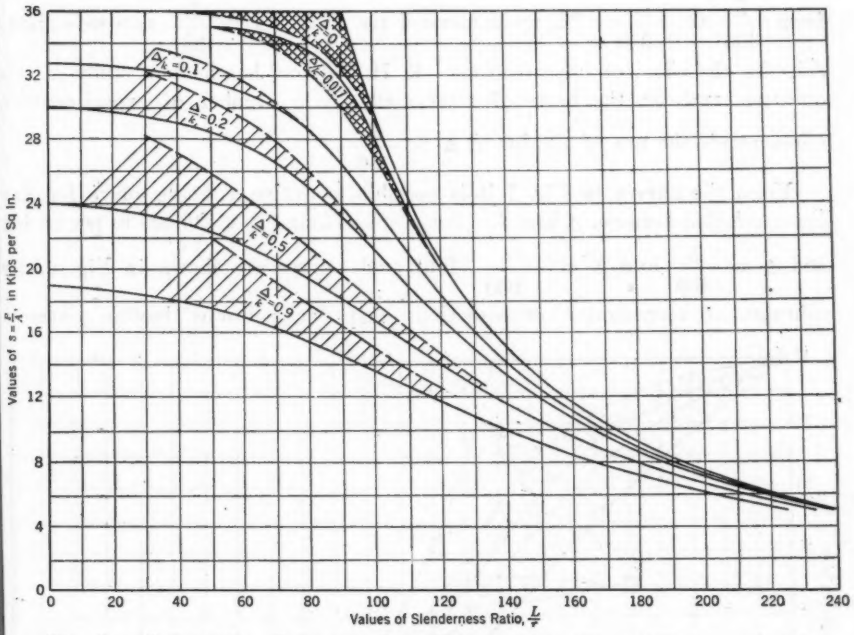

FIG. 8. — COMPARISON BETWEEN YIELD-POINT LOAD AND COMPLETE BUCKLING LOAD; INITIALLY CURVED COLUMN.

the practical design of columns allowance will have to be made for values of $\frac{\Delta}{k}$ greater than 0.1. In general, it may be stated that the difference between the yield-point load and the load for complete collapse is always less for columns than for beams.

Design of a column by the trial and error method may now be made readily by the use of the curves in Fig. 7. The only factor difficult to decide upon is the proper value of the eccentricity ratio, $\frac{\Delta}{k}$, to be used. It has been shown[8] that the effect of any small accidental eccentricity of load on the column can be represented by some definite curvature of the axis. All imperfections of the column are then representable by some "equivalent curvature." Deviation from a straight line, of the axis of the column, due to accidental crookedness undoubtedly increases with the length of the column. It seems

[8] "Columns," by E. H. Salmon, Oxford Technical Publication, p. 152.

logical, therefore, that the eccentricity ratio selected to take care of such imperfections should be some function of the length. The proper function can be determined only on the basis of carefully conducted tests. Professor H. Kayser,[9] by working backward from test results to find the amount of initial curvature which must have been present, found values of Δ ranging from $\frac{L}{400}$ to $\frac{L}{1\,000}$. He recommended the use of $\Delta = \frac{L}{400}$ as a safe allowance for the effect of imperfections. E. H. Salmon[8] has made a study of this question and, on the basis of tests made by a number of experimenters, recommends the use of a value of $\Delta = \frac{L}{750}$.

From the curves in Fig. 7 it is possible to derive a single curve for any given relation between Δ and L. For a steel having $f_y = 36\,000$ lb per sq in., and $\Delta = \frac{L}{400}$ and $\Delta = \frac{L}{1\,000}$, four such curves are shown in Fig. 9, for columns of rectangular cross-sections and for columns having extreme

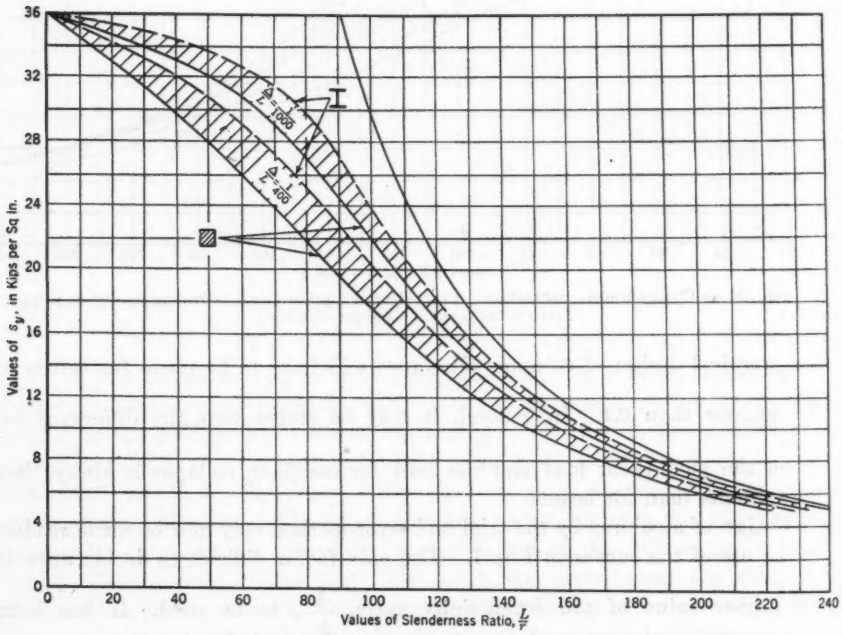

FIG. 9.—PROPOSED DESIGN CURVES FOR PIN-ENDED COLUMNS.

I-sections; that is, having all the material concentrated at the radius of gyration from the axis. Granting that a column of I-section is not likely to have any more initial crookedness than one of rectangular cross-section of the same length, it is seen that the I-section is more efficient in this respect.

[9] "Knickversuche mit doppeltteiligen Rahmenstaben," *Die Bautechnik*, Vol. 12, Berlin. 1930.

For $\Delta = \frac{L}{400}$ the two curves in Fig. 9, plotted with a factor of safety, $n = 2.25$, are shown in Fig. 10 in comparison with standard American specifications for column design as approved by the Society,[3] the American Railway Engineering Association,[3] and the City of Boston, Mass.[3] (The value, 2.25, is selected to make the result comparable with the usual standard formulas based on a working stress of 16 000 lb per sq in.)

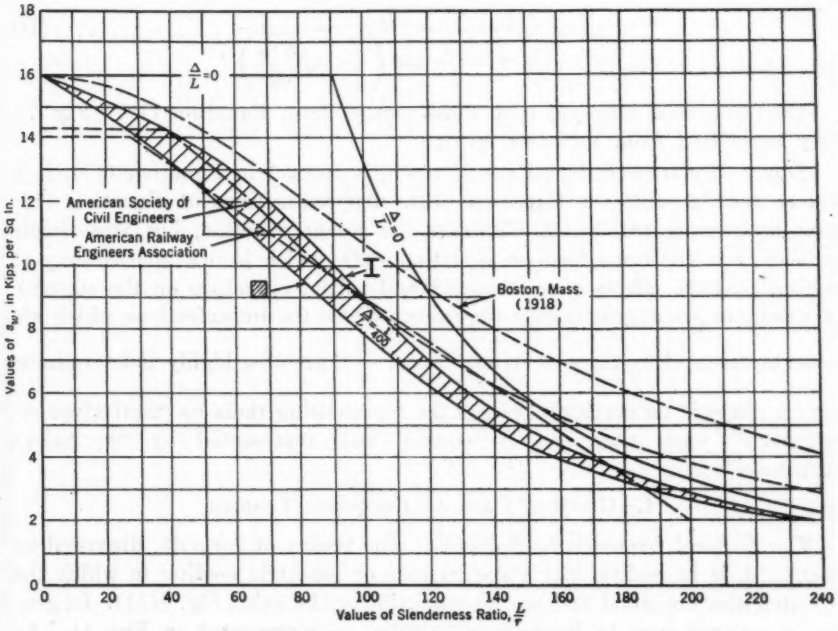

FIG. 10.—PROPOSED DESIGN CURVES COMPARED WITH STANDARD FORMULAS.

3.—IMPERFECTIONS REPRESENTED BY ECCENTRICITY OF LOAD

The effect of slight imperfections in straightness of the axis and central application of load, has been studied in Section 2 on the basis of some definite amount of "equivalent curvature." These imperfections can also be represented by some "equivalent eccentricity" of load on a straight column with a given amount of eccentricity, e. The most serious condition (for direct and bending stresses) will be obtained when the eccentricities are on the same side as shown in Fig. 4(b). For any value of the load, P, the bar assumes a definite shape and the maximum fiber stress, which occurs at the mid-cross-section, is given by the well-known secant formula which may be written in the form:

$$f_m = s\left[1 + \frac{e}{k}\sec\left(\frac{L}{2r}\sqrt{\frac{s}{E}}\right)\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots (14)$$

When the maximum fiber stress is taken equal to the yield point, Equation (14) may be written,

$$s_y = \frac{f_y}{1 + \frac{e}{k} \sec\left(\frac{L}{2r}\sqrt{\frac{s_y}{E}}\right)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (15)$$

In the same manner as before, any desired factor of safety, n, can be incorporated in Equation (14) giving:

$$s_w = \frac{f_y}{n + \frac{n\,e}{k} \sec\left(\frac{L}{2r}\sqrt{\frac{n\,s_w}{E}}\right)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (16)$$

Curves, almost identical with those plotted from Equation (12) (Fig. 7), may be plotted from Equation (15).

Due to the transcendental function which appears in Equation (15), it is not so readily solved as Equation (12). Approximations of Equation (15) have been suggested, which eliminate the secant function, but this simply reduces it to the same form as Equation (12), which is just another way of stating that the effects of eccentricity and initial curvature on the strength of a column are approximately the same. Since the imperfections which are to be represented by the eccentricity ratio, $\frac{e}{k}$, are of a highly indeterminate nature, there is no particular reason for representing them by "equivalent eccentricity" when they can be equally well represented by "equivalent curvature."

4.—General Case of Eccentric Loading

The General Eccentricity Formula.—The secant of formula, discussed in Section 3, is in reality only a special case of eccentric loading in which the eccentricities are equal and on the same side of the axis (Fig. 4(*b*)). In general, a column may be loaded eccentrically, as represented in Fig. 11. In

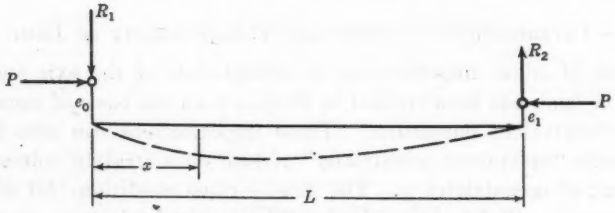

Fig. 11.—General Case of Eccentric Loading.

the diagram the eccentricity, e_0, at the left end is understood to be the larger of the two and is considered as positive, while e_1, the smaller eccentricity, is considered either positive or negative as it is on the same side as e_0, or on the opposite side.

An analysis of this case shows that as long as the load is less than a certain value, which will be denoted by P_q, the maximum bending moment, and

consequently, the maximum fiber stress, occurs at the left end and is simply obtained from the equation (for $P < P_q$):

$$f_m = \frac{P}{A}\left(1 + \frac{e_o}{k}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (17)$$

The value of P_q is given by the equation:

$$s_q = \frac{P_q}{A} = \frac{(\cos^{-1}\alpha)^2\, E}{\left(\frac{L}{r}\right)^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (18)$$

in which, $\alpha = \frac{e_1}{e_o}$, and may vary from + 1 to − 1. Curves representing s_q as a function of $\frac{L}{r}$ for various values of α, can be drawn similar to those shown by broken lines in Fig. 12, which are for $\alpha = + 0.5$ and $\alpha = - 1.0$.

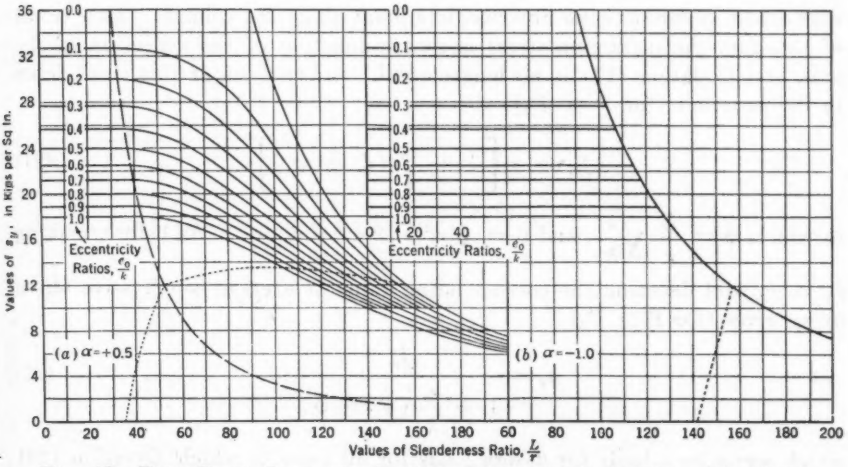

FIG. 12.—YIELD-POINT LOAD FOR VARIOUS ECCENTRICITY RATIOS: (*a*) ECCENTRICALLY LOADED COLUMN (*a*) $\alpha = + 0.5$; (*b*) $\alpha = - 1.0$.

In the case of $\alpha = - 1.0$ (Fig. 12(*b*)), this curve coincides with the Euler curve.

From Equation (17), may be written (for $P < P_q$):

$$s_y = \frac{f_y}{1 + \frac{e_o}{k}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (19)$$

which serves as a basis for design loads, as long as it gives a value for the average compressive stress less than the value given by Equation (18). Looking at this limiting case in another way, by making s_q in Equation (18) equal to s_y in Equation (19) and solving for the corresponding value of $\left[\frac{L}{r}\right]_q$

gives:

$$\left[\frac{L}{r}\right]_q = (\cos^{-1}\alpha)\sqrt{\frac{E}{f_y}\left(1 + \frac{e_o}{k}\right)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (20)$$

Thus, as long as the slenderness ratio of the column is less than the value determined by Equation (20), Equation (19) may be used as a basis for design. It is seen that, especially for negative values of α (bending in a reverse curve), the $\frac{L}{r}$ - range within which Equation (19) may be used, is very large.

This much of the analysis, for loading as represented in Fig. 11, was presented by C. N. Ross, in 1927, who applied it to the case of secondary stresses in truss members.[10]

Referring again to Fig. 11: When the load, P, reaches a value greater than P_q, the section of maximum bending moment, and, consequently, of maximum fiber stress, occurs at some intermediate point along the column. As a result of this fact the maximum stress is now a function of the deflection of the axis, and Equation (17) is no longer valid. An analysis of this case[11] leads to the equation (for $P > P_q$):

$$f_m = s\left[1 + \frac{e_o}{k}(\psi \csc \phi)\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots (21)$$

in which, $\phi = \frac{L}{r}\sqrt{\frac{s}{E}}$, and $\psi = \sqrt{\alpha^2 - 2\alpha\cos\phi + 1}$. As before, making $f_m = f_y$, and denoting the corresponding value of average compressive stress by s_y, gives (for $P > P_q$):

$$s_y = \frac{f_y}{1 + \frac{e_o}{k}(\psi \csc \phi)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (22)$$

which serves as a basis for design loads for all cases to which Equation (19) cannot apply. It will be seen that when $\alpha = +1$ (equal eccentricities on the same side), Equation (22) will reduce to Equation (15).

For various values of $\alpha = \frac{e_1}{e_o}$ curves may be plotted from Equations (19) and (22), which will show the average compressive stress, at which yielding first begins, as a function of the slenderness ratio of the column. For a steel having $f_y = 36\,000$ lb per sq in., and values of $\alpha = +0.5$ and -1.0, characteristic curves are shown in Fig. 12.

For practical purposes, a set of these curves for values of α progressing at intervals of 0.25, between the values of $+1.0$ and -1.0, can be drawn by the

[10] "Equivalent Eccentricities Due to Secondary Stresses," C. N. Ross, *Transactions*, Inst. of Engrs. of Australia, Vol. VIII (1927), p. 8.

[11] See the writer's paper entitled, "Stresses in Eccentrically Loaded Steel Columns," *Publications*, International Assoc. for Bridge and Structural Eng., Vol. 1, Zurich, 1932, p. 507.

successive solution of Equations (19) and (22). Results for intermediate values of α can then be interpolated from these curves.

Allowance for the usual imperfections may be made in this case by the use of an "equivalent eccentricity" rather than "equivalent curvature." These accidental factors may occur in any form, and, therefore, the "equivalent eccentricity" should be chosen in the same direction at each end, since this is the most serious type. Using a value of, say, $e = \frac{L}{400}$, this "equivalent eccentricity" can be determined for any given column. Superposing this value of e at each end on the actual eccentricities, e_0 and e_1, will simply give modified values, e'_0 and e'_1, to be used in determining α. This modified value of α will then be used in selecting the yield-point load from the curves such as those in Fig. 12. The design load is obtained by dividing by the desired factor of safety.

Columns in Rigid Frame Construction.—One of the most important cases of column action to the structural engineer occurs in rigid frame construction, examples of which are shown in Fig. 13. Due to the elastic action of

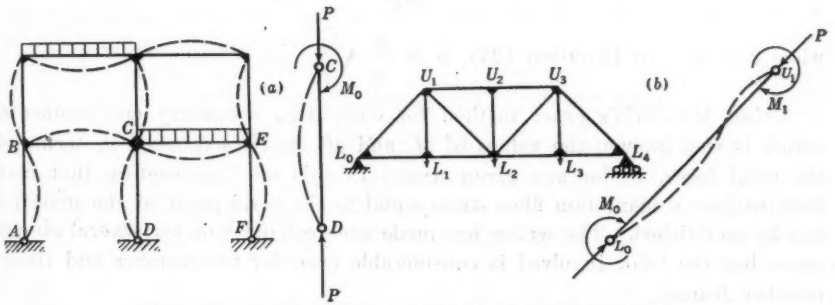

FIG. 13.—THE COLUMN IN RIGID FRAME CONSTRUCTION.

other members meeting at a joint, such a column acts intermediately between pin-ended and completely fixed-ended conditions. The determination of the actual condition requires the use of a theory of the stability of a system of compressed bars. Such a theory has been developed by Mises and Ratzersdorfer.[12] Their treatment, however, is based on the assumption that the material always behaves elastically and it does not determine the stresses that may arise, but rather considers only the problem of the elastic stability of such a system. Furthermore, its application to any system except the simplest becomes extremely laborious.

Since most compression members encountered in structural engineering are comparatively short, the maximum fiber stress is usually the governing factor in design rather than the elastic stability. In order to determine the stress that may occur in a column in a truss, consider the member, $L_0\ U_1$, shown

[12] "Uber die Stabilitätsprobleme der Elastizitätstheorie," von R. v. Mises, *Zeitschrift für angew. Math. u. Mech.*, Vol. 3 (1923), pp. 406–422; also, "Die Knicksicherheit von Rahmentragwerker," von R. v. Mises und J. Ratzersdorfer, *Zeitschrift für angew. Math. u. Mech.*, Vol. 5 (1925), p. 218, and Vol. 6 (1926), p. 181.

in Fig. 13(*b*). In addition to the axial compressive force, P, there will be secondary end moments, M_0 and M_1, arising due to the elastic action of the other members.

An analysis of the maximum bending moment that may occur due to this loading, can be made in exactly the same manner as in the case of the type of loading represented in Fig. 11. The resulting equations for maximum bending moment will be:

$$M_m = M_0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (23)$$

as long as $s < s_q$, in which,

$$s_q = \frac{\left(\cos^{-1}\frac{M_1}{M_0}\right)^2 E}{\left(\frac{L}{r}\right)^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (24)$$

and,

$$M_m = M_0 \ (\csc \phi) \sqrt{\left(\frac{M_1}{M_0}\right)^2 - 2\,\frac{M_1}{M_0}\cos\phi + 1} \quad \ldots\ldots\ldots (25)$$

when $s > s_q$. In Equation (25), $\phi = \frac{L}{r}\sqrt{\frac{p}{E}}$, as before.

Using Manderla's exact method for computing secondary end moments, which is well known, the values of M_0 and M_1 can be expressed in terms of the axial force, P, for any given truss. In this way the loading that may first produce a maximum fiber stress equal to the yield point of the material can be established. The writer has made such calculations for several simple cases, but the labor involved is considerable even for two-member and three-member frames.

It is the usual practice, in computing secondary end moments, to neglect the effect of the axial load, P, on the lateral deflections produced in the bar. It is generally indicated that truss members are usually of such proportion that the effect of the axial force, P, in modifying the secondary end moments, is seldom more than 6% and usually much less.[13] In view of this fact, the excessive labor involved in using Manderla's exact method is scarcely considered justifiable.

Neglecting the effect of the axial force, P, on the deflections (in computing the values of secondary moments) amounts, however, to assuming that these moments increase in direct proportion to the axial force, P; that is, by making this approximation, the effect of the secondary end moments in Fig. 13(*b*) may be represented by the type of loading shown in Fig. 11, in which, e_0 and e_1 will be such that $P \times e_0 = M_0$ and $P \times e_1 = M_1$. Thus, columns in rigid frame construction may be designed on the basis of such curves as those presented in Fig. 12.

[13] "Statically Indeterminate Stresses," by John I. Parcel and George A. Maney, Members, Am. Soc. C. E., John Wiley & Sons, p. 321.

The procedure will be as follows: From the usual approximate method of calculating secondary end moments, the values of M_o and M_1 for each member are determined. Dividing each by the axial force, P, the values of e_o and e_1 are obtained. To make allowance for the effect of initial curvature and imperfections in general these values should be modified, as discussed previously, before computing $\alpha = \frac{e'_1}{e'_o}$ to be used in selecting the average compressive stress, s_y, from the curves (Fig. 12).

In the case of slender members this method of procedure will result in some error (on the side of safety, however). As was mentioned previously, Ross has proposed this same method of procedure for the design of compression members in trusses. He made a study[10] of a large number of existing structures of all types and found that in the great majority of cases the secondary end moments gave values of α such that the member came within the range of Equation (19) rather than Equation (22). Stated in another way, the slenderness ratio of the member was usually less than the value given by Equation (20). In all such cases, of course, the use of the curves (Fig. 12) represents no error at all. In practical cases, when the slenderness ratio is not less than the value determined by Equation (20), the error involved will usually be small and certainly on the side of safety.

PART II.—BUILT-UP STEEL COLUMNS

5.—Effect of Shear in Built-Up Columns

Stability.—Columns, as used in engineering structures, are frequently made of channels laced together by lattice-bars or batten-plates, as shown in Fig. 14. When such a column deflects laterally under load the cross-sections

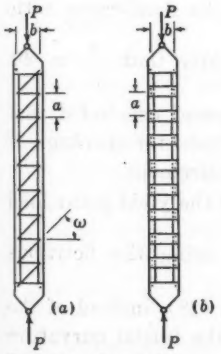

FIG. 14.—BUILT-UP COLUMNS.

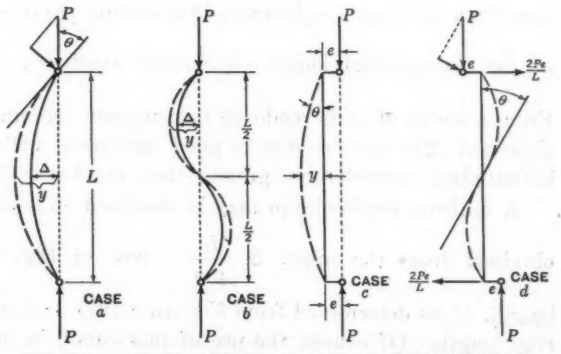

FIG. 15.—POSSIBLE TYPES OF INITIAL IMPERFECTIONS.

of the bar are no longer perpendicular to the end loads, and shearing forces are introduced. In the case of a column having a solid cross-section, the effect of such shearing forces is negligible, but in the case of built-up columns, the effect of the additional distortion due to shear must be taken into account.

An analysis[14] of built-up columns shows that the critical load at which elastic instability occurs can be represented by the Euler equation in terms of a fictitious column having a length, L', thus:

$$P_c = \frac{\pi^2 E I}{(L')^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (26)$$

For the lattice-bar column (Fig. 14(a)), the fictitious length is given by the equation:

$$L' = L\sqrt{1 + \frac{\pi^2 E I}{L^2}\left(\frac{1}{\sin \omega \cos^2 \omega \, E A_d} + \frac{b}{a E A_b}\right)} \quad \ldots\ldots (27)$$

in which, L = the true length of the column; ω = the angle between batten-bar and diagonal; A_d = cross-sectional area of two diagonal bars; A_b = cross-sectional area of two batten-bars; a = unsupported length of channels; and, b = distance between gravity axes of channels. For the batten-plate column (Fig. 14 (b)), the fictitious length is given by the equation:

$$L' = L\sqrt{1 + \frac{\pi^2 E I}{L^2}\left(\frac{a\,b}{12\,E I_b} + \frac{a^2}{24\,E I_c} + \frac{K_2\,a}{0.4\,b\,A_b\,G}\right)} \quad \ldots\ldots (28)$$

in which, a = distance center to center of batten-plates; b = distance between gravity axes of channels; I_c = moment of inertia of one channel about its gravity axis; I_b = moment of inertia of two batten-plates, with respect to gravity axis of bending; A_b = cross-sectional area of two batten-plates; K_2 = a coefficient = 1.2 for rectangular batten-plates; and G = shearing modulus for the battens.

The derivation, upon which these fictitious lengths are based, assumes an infinitely large number of panels. Practically, however, their use will give good results for a very limited number of panels. The specifications of the American Railway Engineering Association require that the slenderness ratio of the unsupported length of channel shall not be greater than $\frac{a}{r_c} = 40$. From a series of tests made on batten-plate columns, Kayser[15] concluded that Equation (26) gave results in good agreement with the tests for spacings of batten-plates considerably greater than the foregoing requirement.

A built-up steel column may be designed on a basis of the yield-point load obtained from the upper $\Delta \frac{L}{400}$-curve in Fig. 9 by using the fictitious length, L', as determined from Equation (27) or Equation (28), instead of the true length. Of course, the use of this curve, in which the initial curvature to represent the effect of imperfections has been taken as a function of the length, will amount to assuming the imperfections as a function of the fictitious rather than the true length. This will only result in giving a slight

[14] "Strength of Materials," S. Timoshenko, Pt. II (Van Nostrand), p. 592.

[15] "Knickversuche mit doppelteiligen Rahmenstaben," von H. Kayser, *Bautechnik*, Vol. 12, 1930.

additional factor of safety which is probably not out of order for built-up columns.

Effect of Imperfections on Shearing Stresses.—It is present-day practice to design the lacing or batten-bars, empirically, to resist possible shearing forces. For example, the A.R.E.A. Specifications require that the lacing-bars shall be designed to resist shearing forces not less than 0.025 times the total compressive force on the column.

In discussing the design of lattice-bars and battens to resist possible shearing forces, it seems logical to proceed as before and assume, as a basis of computation, an imperfection in the form of initial curvature or eccentricity of load. When this has been selected, it will be possible to evaluate the maximum shearing force that arises for any value of the compressive load, P. This maximum shearing force will then be calculated for the value of the load, P_y, which first causes yielding in the extreme fibers. The shearing force thus calculated will be used as a basis for design. In this way the strength of the column in shear will be consistent with its strength in bending and thrust.

The imperfections should now be taken in the most serious form, as far as shearing forces are concerned. Possible types of such initial imperfections are represented in Fig. 15. Further consideration of some of these initial conditions to represent imperfection can be ruled out.

First, consider the imperfections as represented by the half wave of a sine curve shown as Case a in Fig. 15(a). For any value of the load, P, there will be a definite value of the angle, θ, at the end, and the maximum shearing force occurring here will be, $V = P \sin \theta$. In the second case, when the initial curvature is taken as a full sine curve (S-shaped), it will be logical to assume Δ as the same function of $\frac{L}{2}$ instead of L, and each half of the column will be identical with the previous case. However, since the column will buckle in one wave (regardless of its initial shape) at the Euler load for the length, L, this case can never give rise to as great shearing forces as Case a. Case c, likewise, will never be as serious as Case a because, at the load producing failure, the maximum bending moment at the center must be approximately the same for each case. In Case a this moment is $P \times y$, while, in Case c, it is $P\ (y + e)$; hence, y (and, consequently, θ, at the end) will always be less for the eccentrically loaded column. If θ is less, the shear, $P \sin \theta$, will be less.

Case d only, then, needs further consideration. When the eccentricities are chosen on opposite sides of the axis, definite horizontal reactions, $\frac{2Pe}{L}$, arise at the ends. Remembering that θ will be a small angle, it is seen from Case d, Fig. 15, that the shearing force will be a maximum at the center cross-section and equal to $\frac{2Pe}{L} + P \sin \theta$. It is possible that this condition

may introduce greater shearing forces than will initial curvature. Cases *a* and *d*, in Fig. 15, will now be considered in more detail.

6.—Shear Due to Imperfections Represented by Initial Curvature

Considering Case *a*, Fig. 15, the equation of the elastic line is represented by Equation (6). Taking the first derivative of this equation with respect to x will give the slope at any cross-section as,

$$\frac{dy}{dx} = \frac{\Delta \pi^2}{\pi^2 - q^2 L^2} \frac{\pi}{L} \cos \frac{\pi x}{L} \quad \ldots\ldots\ldots\ldots\ldots (29)$$

This will be a maximum at the end, where $x = 0$, and hence,

$$\tan \theta_m = \frac{\Delta \pi^2}{\pi^2 - q^2 L^2} \frac{\pi}{L} \quad \ldots\ldots\ldots\ldots\ldots\ldots (30)$$

Substituting the values, $q^2 = \frac{P}{EI}$ and $P_e = \frac{\pi^2 EI}{L^2}$, Equation (30) may be written,

$$\tan \theta_m = \frac{\Delta \pi}{L} \frac{1}{1 - \frac{s}{s_e}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (31)$$

Since the maximum shearing force is $V = P \sin \theta_m = P \tan \theta_m$ (for small angles), Equation (31) may be written:

$$v = \frac{V}{A} = s \frac{\Delta \pi}{L} \frac{1}{1 - \frac{s}{s_e}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (32)$$

For any given slenderness ratio, $\frac{L}{r}$, the value of s, at which yielding first begins, is given by Equation (12). For a fixed value of $\frac{L}{r}$, this formula may be considered as representing the relation between the average compressive stress, s_y, for yielding, and the amount of initial curvature as represented by Δ. Solving Equation (12) for Δ and substituting the value obtained into Equation (32) gives,

$$v_y = \frac{\pi (f_y - s_y)}{\frac{L}{k}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (33)$$

Equation (33) gives the maximum value of the average shearing stress which the details are called upon to resist when the column is at its yield-point load, and, consequently, represents the desired basis for design.

When $\frac{L}{k} = 0$, $s = f_y$, and Equation (33) is seen to become indeterminate. To evaluate v_y for this case, $\frac{\Delta}{L}$ in Equation (32) may be made equal to the

chosen ratio (for example, $\frac{\Delta}{L} = \frac{1}{400}$) for which Equation (32) becomes,

$$v = \frac{V}{A} = s \frac{\pi}{400} \frac{1}{1 - \frac{s}{s_e}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (34)$$

from which, the value of v_y for $\frac{L}{k} = 0$ is seen to be, $v_y = f_y \frac{\pi}{400}$.

It is interesting to note in this connection that while the effects of initial curvature and eccentricity of load on the bending stresses are approximately the same, this is not the case with regard to shearing stresses. Initial curvature is considerably more serious for shear, due to the fact that to begin with there is some definite slope at the end, while in the straight bar, eccentrically loaded, this slope at the end must be built up by the load.

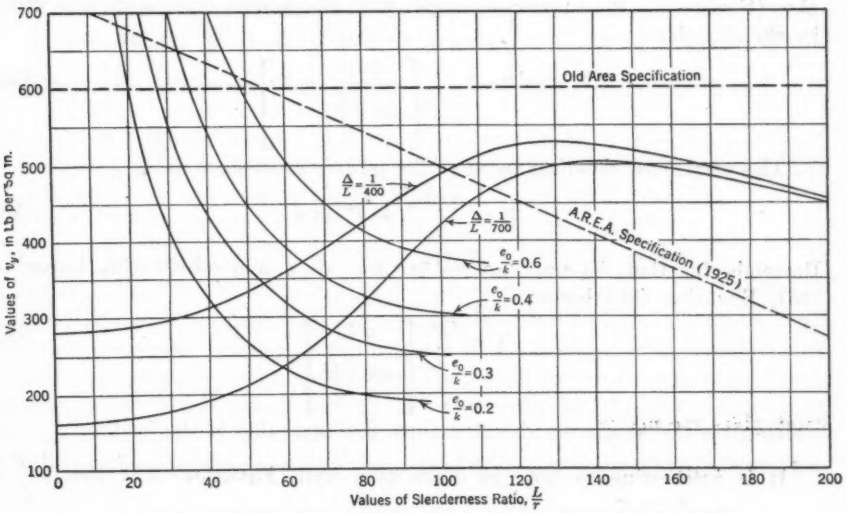

FIG. 16.—AVERAGE SHEAR STRESS DUE TO VARIOUS IMPERFECTIONS.

Taking values of s_y from the curves in Fig. 9, in which $k = r$, values of v_y may be computed from Equation (33) for various values of $\frac{L}{r}$, and any given ratio of $\frac{\Delta}{L}$. In this manner, curves may be plotted showing how the maximum average shearing stress, v_y, varies with the slenderness ratio, $\frac{L}{r}$, for any given ratio of $\frac{\Delta}{L}$. Such curves are shown in Fig. 16 for $\frac{\Delta}{L} = \frac{1}{400}$ and $\frac{\Delta}{L} = \frac{1}{700}$

From an examination of these curves it is seen that the maximum shear occurs for columns having a slenderness ratio of about 130, which is greater than the usual allowance for slenderness made by specifications. For columns of the usual proportions the maximum average shearing stress varies from about 300 to 550 lb per sq in. in the case of the more severe allowance for imperfections. In Fig. 16, for purposes of comparison, a curve is shown which represents the A.R.E.A. Specifications of 1925 and a curve representing an older A.R.E.A. Specification. It might be judged that the new specification was unsafe for slender columns, but it must be remembered that this same specification limits the slenderness ratio of main compression members to 100 or less.

7.—Shear Due to Imperfections Represented by Eccentricity

Considering imperfections represented in the form of eccentricity of load, as shown in Fig. 15 (*d*) and proceeding in the same manner as in the previous case (Section 6), the maximum angle, θ, at the middle cross-section is given by the equation:

$$\tan \theta_m = \frac{e}{L}\left[\frac{q\,L}{\sin \frac{q\,L}{2}} - 2\right] \qquad (35)$$

The maximum shearing force at the middle cross-section is,

$$V = \frac{2\,P\,e}{L} + P \sin \theta \qquad (36)$$

Remembering that, for small angles, $\tan \theta = \sin \theta$, and substituting Equation (35), Equation (36) becomes,

$$V = \frac{P\,e}{L}\left[\frac{q\,L}{\sin \frac{q\,L}{2}}\right] \qquad (37)$$

which gives the maximum shearing force for any value of the load.

It is interesting to note in connection with Equation (37) that $\frac{2\,P\,e}{L}$ represents the shearing force at all cross-sections if deformation of the bar is neglected. Assuming that $q = \sqrt{\frac{P}{E\,I}}$ is very small, the term in the brackets in Equation (37) becomes equal to 2. However, as the load, P, is increased, this factor increases until when P reaches the Euler value it becomes equal to π. That is, for slender columns, where the load may reach the Euler value before the maximum fiber stress reaches the yield point, the shearing force may become 57% greater than the value that would be obtained by neglecting deformations.

Considering Equation (37) again, and dividing both sides by A, will give:

$$v = s\,\frac{e}{L}\left(\frac{q\,L}{\sin \frac{q\,L}{2}}\right) \qquad (38)$$

In this case where the effect of imperfections is being represented by the particular type of eccentricity represented in Fig. 15(*d*), it will be more reasonable to use a constant value of e than to take it as a function of the length, since for a short column there is just as much chance for accidental eccentricity of load as for a long one.

For this case, $\alpha = -1$, and the value of s at which yielding begins, is given by Equation (19). Solving Equation (19) for the eccentricity, e, and substituting the value obtained into Equation (38) gives,

$$v_y = \frac{f_y - s_y}{\frac{L}{k}} \left(\frac{qL}{\sin \frac{qL}{2}} \right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (39)$$

Comparing Equation (39) with Equation (33), it is seen that for slender columns they give about the same result. However, for short columns, Equation (39) may give considerably higher values for the average shearing stress.

In Fig. 16 are shown four curves plotted from Equation (39), assuming an extreme I-section ($k = r$), for eccentricity ratios, $\frac{e}{k} = 0.2$, 0.3, 0.4, and 0.6. From a general study of all the curves in Fig. 16, a constant allowance for shearing stresses of 600 lb per sq in., in accordance with the old A.R.E.A. Specifications, would seem to be justifiable.

8.—Shear in the General Case of Eccentric Loading

Shear Stress at Yield-Point Load.—From an examination of the curves in Fig. 16 for equal eccentricities on opposite sides of the axis, it is seen that, for short columns, the shear arising due to such loading may be considerable. In the case of short stocky members in rigid frame construction, where large secondary end moments may arise, it seems advisable to know something about the possible extent of such shearing forces. The brief analysis[16] made herein will be for the equivalent case of eccentric loading, as considered in Section 4.

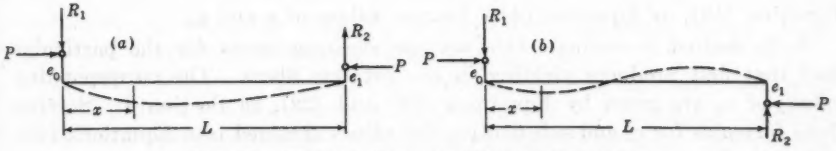

Fig. 17.—General Case of Eccentric Loading.

Consider the pin-ended column with eccentrically applied end loads, as shown in Fig. 17. The eccentricity, e_o, is taken as the larger so that in Fig. 17(*a*), $\alpha = \frac{e_1}{e_o} > 0$, and in Fig. 17(*b*), $\alpha = \frac{e_1}{e_o} < 0$. By the ordinary pro-

[16] For a detailed analysis, see the writer's paper, "Shearing Stresses in Steel Columns," *Publications*, International Assoc. for Bridge and Structural Eng., Vol. II, Zurich, 1934, p. 480.

cedure the slope of the elastic line at any point, distant x from the left end, is found to be:

$$\frac{dy}{dx} = -q\,e_o \sin q\,x + q\,e_1 \frac{\cos q\,x}{\sin q\,L} - q\,e_o \frac{\cos q\,x}{\tan q\,L} + \frac{e_o}{L} - \frac{e_1}{L} \quad \ldots\ldots(40)$$

For $\alpha > 0$ (Fig. 17(a)) the maximum shearing force will occur at the end of the column where $x = L$, and will be:

$$V = \frac{P(e_o - e_1)}{L} + P\left[\frac{dy}{dx}\right]_{x=L} \quad \ldots\ldots\ldots\ldots\ldots\ldots(41)$$

For $\alpha < 0$ (Fig. 17(b)), the maximum shearing force will occur at the inflection point some place within the column and will be,

$$V = \frac{P(e_o - e_1)}{L} + P\left[\frac{dy}{dx}\right]_{max} \quad \ldots\ldots\ldots\ldots\ldots\ldots(42)$$

Substituting the values of $\left[\frac{dy}{dx}\right]_{x=L}$ and $\left[\frac{dy}{dx}\right]_{max}$ into Equations (41) and (42), respectively, and dividing through by A, gives:

For $\alpha > 0$:

$$v = s\,\frac{e_o}{L}\,\phi\,\beta\,\csc\,\phi \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(43)$$

and for $\alpha < 0$:

$$v = s\,\frac{e_o}{L}\,\phi\,\psi\,\csc\,\phi \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(44)$$

in which, $\phi = q\,L = \frac{L}{r}\sqrt{\frac{s}{E}}$; $\beta = 1 - \alpha\cos\phi$; and $\psi = \sqrt{\alpha^2 - 2\,\alpha\cos\phi + 1}$.

For any given value of $\alpha = \frac{e_1}{e_o}$ the average shearing stress may be computed from Equation (43), or Equation (44), for any values of s and e_o.

It is desired to evaluate this average shearing stress for the particular load that first produces yielding in the extreme fibers. The corresponding values of s_y are given by Equations (19) and (22), in Section 4. Solving these formulas for e_o and substituting the values obtained into Equations (43) and (44) gives:

$$v_y = \frac{(f_y - s_y)\,F}{\frac{L}{k}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(45)$$

in which, F is a numerical factor as follows:

For $\alpha > 0$, and $P < P_q$:

$$F = \phi\,\beta\,\csc\,\phi \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(46)$$

For $\alpha > 0$, and $P > P_q$:

$$F = \frac{\phi \beta}{\psi} \quad \ldots\ldots\ldots\

For $\alpha > 0$, and $P > P_q$:

$$\frac{f_y - 3s}{f_y - s} \alpha \phi \sin \phi \left[\frac{1}{\psi^2} - \frac{1}{\beta}\right] \quad \text{.....(51)}$$

For $\alpha < 0$, and $P < P_q$:

$$\frac{f_y - 3s}{f_y - s} = \phi \left[\cot \phi - \frac{\alpha \sin \phi}{\psi^2}\right] \quad \text{.....(52)}$$

and for $\alpha < 0$, and $P > P_q$:

$$s = \frac{f_y}{3} \quad \text{.....(53)}$$

Equations (50) to (53) may be solved graphically and the resulting s - criteria for absolute maximum shear shown by dotted curves such as those in Fig. 12.

When the criteria call for $s = 0$ (Fig. 12) this must be interpreted as meaning an infinitely small axial force at an infinitely large eccentricity. In other words, for short columns, bending by reverse couples at the ends is the worst possible type of loading for shearing stresses. It is interesting to note that for the particular steel chosen ($f_y = 36\,000$ lb per sq in.), these curves intersect the Euler curve at $\frac{L}{r} = 157$. This means that in the case of slender columns the worst possible type of loading for shear is an axial load without any eccentricity. For the intermediate columns the criteria require some combination of thrust and bending to produce the worst conditions for shear.

Using the values of s from these curves in Equation (45), the curves of absolute maximum shear for the various values of α considered, have been plotted in Fig. 18. In each diagram, this curve of absolute maximum shear stress is shown by the dotted line.

The absolute maximum shear runs fairly high, especially for negative values of α. Therefore, in the case of short compression members in rigid frames, it would seem more advisable to use the actual values of e_o and α obtained by secondary stress calculations in determining the shear stress, than to try to make some blanket allowance for all cases. The usual additional allowance for the effect of imperfections should be made in the same manner as outlined in Section 4. In this case, however, eccentricities on opposite sides of the axis to represent imperfections would be superposed on the actual eccentricities, e_o and e_1, to give the modified values to be used in determining α.

Acknowledgment

The writer is particularly indebted to Professor S. Timoshenko, of the University of Michigan, for invaluable help and advice in the preparation of this paper.

Summary

Although, in this paper, the writer does not pretend to solve, completely, the problem of column design, he offers what is believed to be a more logical basis of procedure than is represented by present-day practice. It is recom-

mended that, as a basis of calculation, some imperfection in the form of an initial curvature or eccentricity of load be taken, and then that the working load be selected on a basis of the load first producing yielding in the extreme fibers due to the assumed form of imperfection.

For pin-ended columns the final proposal for a basis of design is represented by the curves for $\Delta = \frac{L}{400}$ in Fig. 9. For columns in rigid frame construction, in which secondary end moments arise due to the rigidity of the joints, it is proposed to use a set of curves such as those illustrated by Fig. 12 as a basis of design, in which the values of α and e_o will be determined by the usual approximate method of calculating secondary stresses.

For pin-ended columns, which are built up with lacing or battens, an average shearing stress of 600 lb per sq in. is proposed as a basis for the design of the lacing or battens.

For built-up columns in rigid frame construction, it is proposed to use curves such as as those presented in Fig. 18, as a basis for the design of lacing and battens, in which the values of α and e_o will be found from the secondary stress calculations.

APPENDIX

NOTATION

The following symbols conform as nearly as practicable with the "Symbols for Mechanics, Structural Engineering, and Testing Materials", advanced by the American Standards Association[4]. Some of the symbols do not appear in the paper, but are included herein, as a matter of information, and for the guidance of discussers:

a = distance, center to center, of column stiffeners (batten-plates or lattice-bars); also, a = unsupported length of channels. As a subscript, a denotes "average".

b = distance between gravity axes of channels; as a subscript, b denotes "batten-bars".

c = distance from neutral axis to extreme fiber; as a subscript, c denotes "critical".

d = depth; as a subscript, d denotes "diagonal bars".

e = initial eccentricity of load; as a subscript, e denotes "Euler".

f = unit stress; f_y = yield-point stress.

i = any integer; as a subscript, i denotes 'initial".

k = core radius of a cross-section $= \frac{S}{A}$.

m = Poisson's ratio; as a subscript, m denotes "maximum".

n = a factor of safety.

q = a substitution factor $= \sqrt{\frac{P}{EI}}$; as a subscript, q denotes "a certain value".

r = radius of gyration $= \sqrt{\frac{I}{A}}$.

$s = \frac{P}{A}$ = average compressive stress on the column; s_y = average compressive stress at which yielding in the extreme fibers first begins; s_w = allowable working stress; $s_e = \frac{\pi^2 E}{\left(\frac{L}{r}\right)^2}$ = stress corresponding to the Euler load; and s_q = a certain value of s.

t = thickness.

v = average shearing stress, $\frac{V}{A}$; v_y = average shearing stress when Load P is at its yield-point value.

w = load per unit distance; as a subscript, w denotes "working load".

x = variable co-ordinate of Deflection y.

y = deflection of a column, distant x from one end; y_o = initial deflection of a deflected column; as a subscript, y denotes "yield point".

A = area; A_d = cross-sectional area of two diagonal bars; A_b = cross-sectional area of two batten-bars.

C = a constant to be evaluated from the end conditions of a bar.

E = modulus of elasticity.

F = a numerical factor (Equation (45)).

G = shearing modulus for batten-plates.

I = rectangular moment of inertia with respect to gravity axis of bending; $I_c = I$ for one channel, about its gravity axis; $I_b = I$ of two batten-plates with respect to gravity axis of bending.

K = a coefficient = 1.2 for rectangular batten-plates.

L = length; L' = a fictitious column length.

M = bending moment; M_o and M_1 = secondary end moments.

P = compressive axial load on a column; P_e = the Euler load for a pin-ended column $= \frac{\pi^2 EI}{L}$; P_c = critical load for any column; P_y = load that first produces yielding, due to eccentricity; P_w = allowable working load.

S = section modulus.

V = total shear force at any cross-section.

W = total load.

α = a ratio, $\frac{e_1}{e_o}$.

$\beta = 1 - \alpha \cos \phi$.

Δ = maximum initial deflection of an initially curved column.

θ = an angle (Fig. 16).

$\pi = 3.1416$.

$\phi = q\,L = \frac{L}{r}\sqrt{\frac{s}{E}}$.

$\psi = \sqrt{\alpha^2 - 2\,\alpha \cos \phi + 1}$.

ω = angle between batten-bar and diagonal bar.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

PAPERS

A DIRECT METHOD OF MOMENT DISTRIBUTION

By T. Y. Lin,[1] Jun. Am. Soc. C. E.

Synopsis

The Cross method of distributing fixed-end moments in the analysis of continuous frames[2] has become recognized as such a convenient tool of structural design that it is unnecessary to state its importance, or its advantages over the older, classical methods. However, the necessity of going through a series of approximations is still disliked by many engineers.

The purpose of this paper is to present an "exact" method[3] of moment distribution, derived from the fundamental conceptions of the original method, but applied without the series of approximations. Designers of continuous frames will find that the method, being a direct corollary of the moment-distribution principle, is easy to learn. The time saved is considerable as compared with other methods, especially for frames subject to several conditions of loading, frames requiring the use of influence lines, frames with members of variable section, and frames in which convergence is slow by the original method and in which greater accuracy is desired. This method also affords a direct visualization of the "transfer" of moments in continuous frames, and thus throws new light on the economy of proportioning, the effect of fixation at the supports, of continuity, etc., which is true of the original method only indirectly.

Definitions and Notation

The following notation is used: K = stiffness of an end of a member $= \frac{I}{L}$; γ = carry-over factor of an end of a member; M' = fixed-end moment at an end of a member; and, M = actual moment at an end of a member.

Note.—Discussion on this paper will be closed in **March, 1935**, *Proceedings*.

[1] With Ministry of Railways, Chinese National Govt., Nanking, China.

[2] "Analysis of Continuous Frames by Distributing Fixed-End Moments", by Hardy Cross, M. Am. Soc. C. E., *Transactions*, Am. Soc. C. E., Vol. 96 (1932), pp. 1–156.

[3] A more complete treatment is given in a thesis by T. Y. Lin, presented to the Univ. of California in partial fulfillment of the requirements for the degree of Master of Science in Civil Engineering.

The subscripts have the following significance: ab denotes End A of Member AB; ba denotes End B of Member AB; $b2$ denotes End B of Member $B2$; dc denotes End D of Member CD, etc.; and m denotes a quantity "modified", due to the actual restraint of the support. For example, K_{ab} = stiffness of End A in Member AB; and γ_{sbm} = a modified carry-over

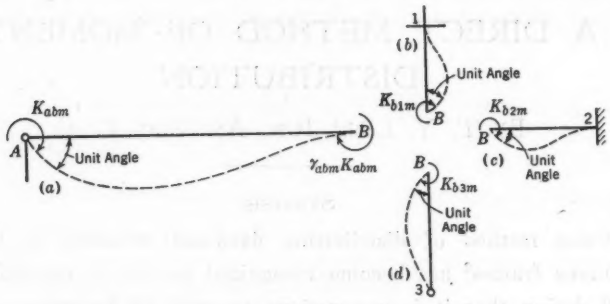

FIG. 1

factor of End 3 in Member $B3$, or the moment at End B due to a unit moment at End 3 with End B under the actual condition of restraint. Furthermore, (see Fig. 1), let,

$$R_{ba} = \frac{K_{ba} + K_{b1m} + K_{b2m} + K_{b3m} + \cdots}{K_{ba}} = \frac{K_{ba} + \Sigma K_{bnm}}{K_{ba}} \quad \ldots\ldots(1)$$

In Fig. 2, Member AB is simply supported at End A and fixed at Point B. Apply a moment at A to produce a unit rotation at that end of the Mem-

FIG. 2

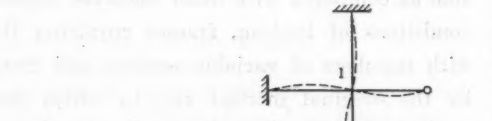

FIG. 3

ber AB. Then, by definition, K_{ab} = the moment applied at End A = stiffness at End A of Member AB; $\gamma_{ab}\,K_{ab}$ = the moment induced at End B due to a stiffness, K_{ab} at End A; and γ_{ab} = the carry-over factor, from End A to End B.

In Fig. 3, Member AB is simply supported at End A and is connected to several members at Joint B—$B1$, $B2$, $B3$, etc. Apply a moment at A to produce a unit rotation at that end of the member. Then, K_{abm} = the moment applied at A = the modified stiffness at End A of Member AB,

with End B held by its restraining members; γ_{abm} K_{abm} = the moment induced at End B in Member AB, due to K_{abm} at End A; and γ_{abm} = the modified carry-over factor at End A.

Sign Conventions

A moment at an end of a member is considered positive when the external moment acting at that end is clockwise. Therefore, a moment carried over does not change in sign; and the algebraic sum of all the end moments at any joint equals the external moment applied at that joint, which is usually zero. (See Fig. 4.)

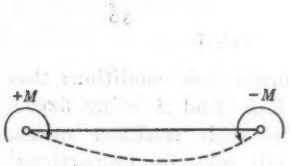

FIG. 4

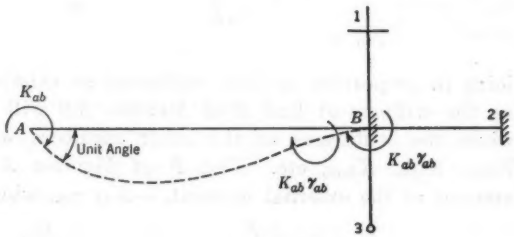

FIG. 5

Theory and Derivation of Equations

In the original method of moment distribution, the stiffness and the carry-over factor of an end of a member are calculated on the assumption that the other end is fixed. Hence, it is necessary to release one joint at a time, holding the joints at the other end of the connected members fixed, in order that the unbalanced moment at a joint may be distributed to the connected members in proportion to their stiffnesses and the moments may be re-proportioned and transferred by the carry-over factors. In the method herein presented, however, the modified values of the stiffness and carry-over factor of each end of each member are calculated in accordance with the actual condition of restraint at the other end. Hence, it is possible to release all the joints simultaneously and to distribute the unbalanced moments once only.

Only two formulas are necessary for the application of this method: One for the value of K_{abm}, the other for the value of γ_{abm}. These formulas will be derived from the fundamental concepts of the moment-distribution method as follows:

Given: The values, K_{ab}, K_{ba}, γ_{ab}, and γ_{ba}, of Member AB, and the values of K_{bnm} (see Equation (1)) for the members connected to AB at End B.

Required: The modified stiffness, K_{abm}, and the modified carry-over factor, γ_{abm}, for End A of Member AB.

Solution:

Step 1.—In Fig. 5, fix Joint B. Apply a moment at End A equal to K_{ab} to produce a unit rotation of that end. Since End B is fixed, a carry-over moment will be produced at End B of Member AB equal to $K_{ab}\ \gamma_{ab}$, which is held in equilibrium by the external moment, $K_{ab}\ \gamma_{ab}$.

Step 2.—Fix End A in this rotated position, as shown in Fig. 6. Release Joint B by applying an external moment of $-K_{ab}\ \gamma_{ab}$, which will rotate all the members meeting at that joint through an equal angle. Hence, this external moment will be distributed among the members meeting at the

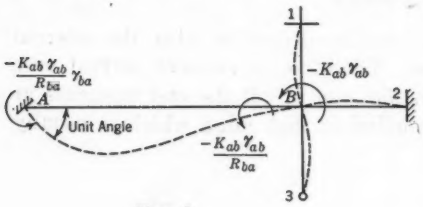

FIG. 6

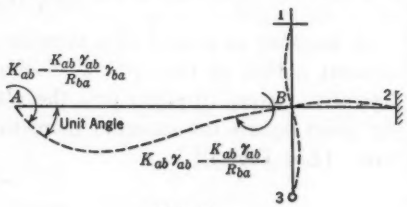

FIG. 7

joint in proportion to their stiffnesses as existing under this condition; that is, the stiffness at End B of Member AB will be K_{ba} (End A being fixed), while the stiffnesses of the other members will be their modified values, K_{b1m}, K_{b2m}, K_{b3m}, etc. End B of Member AB will take its proportional amount of the external moment, $-K_{ab}\ \gamma_{ab}$, which is,

$$-K_{ab}\ \gamma_{ab}\left(\frac{K_{ba}}{K_{ba}+K_{b1m}+K_{b2m}+K_{b3m}+\ldots}\right)$$

$$=-K_{ab}\ \gamma_{ab}\left(\frac{K_{ba}}{K_{ba}+\Sigma K_{bnm}}\right)$$

or,

$$M=\frac{-K_{ab}\ \gamma_{ab}}{R_{ba}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Since End A is fixed, the distributed moment (Equation (2)), of End B will be carried over to End A by the carry-over factor, γ_{ba}. Hence, a moment will be produced at End A of Member AB equal to $\frac{-K_{ab}\ \gamma_{ab}}{R_{ba}}\gamma_{ba}$.

Step 3.—Next, examine the resulting elastic and static condition of Member AB at the end of Steps 1 and 2. (See Fig. 7.) End A of the member is rotated a unit angle and End B is connected to its adjoining members without any external restraint. The resulting moment at End A is the sum of the moments produced in the two steps; that is, $K_{ab}-\frac{K_{ab}\ \gamma_{ab}}{R_{ba}}\gamma_{ba}$. This is the moment at A necessary to produce a unit rotation at End A of Member AB, with End B under its actual condition of restraint. By definition, it is the modified stiffness at End A of Member AB. Therefore,

$$K_{abm}=K_{ab}-\frac{K_{ab}\ \gamma_{ab}}{R_{ba}}\gamma_{ba}=K_{ab}\left(1-\frac{\gamma_{ab}\ \gamma_{ba}}{R_{ba}}\right) \qquad \ldots\ldots (3)$$

The resulting moment at End B of Member AB, $K_{ab}\ \gamma_{ab}-\frac{K_{ab}\ \gamma_{ab}}{R_{ba}}$, is that produced at that end due to a unit rotation of End A, with End B

under its actual condition of restraint. Its ratio to the resulting moment applied at A is, by definition, the modified carry-over factor at that end of the member, AB. Therefore,

$$\gamma_{abm} = \frac{K_{ab}\,\gamma_{ab} - \frac{K_{ab}\,\gamma_{ab}}{R_{ba}}}{K_{ab}\left(1 - \frac{\gamma_{ab}\,\gamma_{ba}}{R_{ba}}\right)} = \gamma_{ab}\,\frac{R_{ba} - 1}{R_{ba} - \gamma_{ab}\,\gamma_{ba}} \quad \ldots\ldots\ldots(4)$$

Equations (3) and (4) are the only ones necessary in this method. They may be simplified for special cases, as follows: (1) For a fixed end, R_{ba} equals infinity; (2) for a hinged end, R_{ba} equals unity; (3) for members of uniform moment of inertia, $\gamma_{ab}\,\gamma_{ba} = \frac{1}{4}$; and (4) for symmetrical members, $\gamma_{ab} = \gamma_{ba}$.

Application of Method

The general procedure consists of two distinct steps: First, to analyze the structure under no load; that is, to find the K_m and γ_m-values; and, second, to distribute the unbalanced fixed-end moments.

Step 1.—Find the values of R, K, and γ of both ends of all members by the usual methods. When the R-value of one end of a member is known, the K_m and γ_m-values of the other end can be found from Equations (3) and (4), respectively. For a hinged end, R is unity; for a fixed end, R is infinity; and, for all other cases, R will have values between the two, and can be either calculated or estimated from the modified stiffnesses of the connected members.

Step 2.—Find the fixed-end moments of each loaded or deformed member by the usual methods. Distribute the unbalanced moments at each joint to all the members meeting at the joint in proportion to their modified stiffnesses. Carry over the distributed moment of each member to the other end by the corresponding modified carry-over factor. Distribute each carried-over moment among the connected members in proportion to their modified stiffnesses. Continue this process of carrying over and distributing to the supports of the structure. Add the fixed-end moment, the distributed moments, and the carried-over moments, to find the resulting moment at each end of each member.

Steps 1 and 2 explain the general procedure only. They can be modified easily to suit individual cases. Some of the important variations will be shown in the subsequent illustrations.

Limitation of the Method

Since the two fundamental equations (Equations (3) and (4)) are derived from the first principles of the moment-distribution method, this modified method has the same limitations as the original. Problems such as sidesway,[4] that can be solved by the original method, can be solved similarly by the writer's method.

[4] *Transactions*, Am. Soc. C. E., Vol. 96 (1932), p. 9.

Examples

Example 1.—Consider the frame shown in Fig. 8. Ends *G* and *F* are hinged; Ends *H* and *J* are fixed; and Ends *A* and *B* are connected to members above the frame (not shown in the diagram), such that End *A* of Member *AD* has a value of $R_{ad} = 3.00$ and End *B* of Member *BE* has a value of $R_{be} = 6.00$.

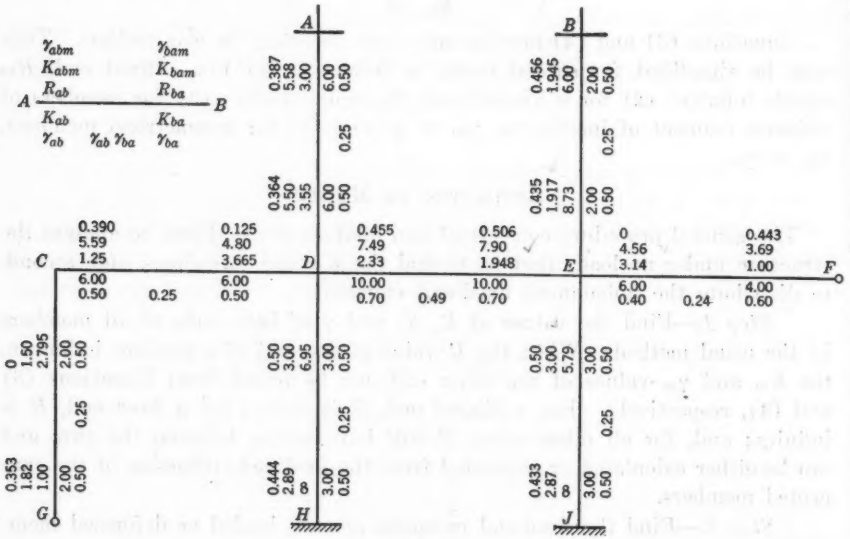

FIG. 8

Step 1.—The *R*, *K*, and γ -values of all the members are determined first from Equations (3) and (4), and arranged for each member as shown in the key sketch at the top of Fig. 8. In Member *GC*, End *G* is hinged and $R_{gc} = 1$. Then, from Equation (3),

$$K_{cgm} = K_{cg}\left(1 - \frac{\gamma_{cg}\,\gamma_{gc}}{R_{gc}}\right) = 2\left(1 - \frac{\frac{1}{2} \times \frac{1}{2}}{1}\right) = 1.50$$

and, from Equation (4),

$$\gamma_{cgm} = \gamma_{cg}\,\frac{R_{gc} - 1}{R_{gc} - \gamma_{gc}\,\gamma_{cg}} = \frac{1}{2}\,\frac{1 - 1}{1 - \frac{1}{4}} = 0$$

Similarly, in Member *CD*, $R_{cd} = \dfrac{6 + 1.5}{6} = 1.25$; $K_{dcm} = 6\left(1 - \dfrac{0.25}{1.25}\right)$ $= 4.80$; and, $\gamma_{dcm} = \dfrac{1}{2}\,\dfrac{0.25}{1.00} = 0.125$. In the case of Member *HD*, R_{hd} = infinity; $K_{dhm} = K_{dh} = 3$; and, $\gamma_{dhm} = \gamma_{dh} = \frac{1}{2}$. In Member *AD*, $R_{ad} = 3$; $K_{dam} = 6\left(1 - \dfrac{0.25}{3}\right) = 5.50$; and $\gamma_{dam} = \dfrac{1}{2}\left(\dfrac{3 - 1}{3 - 0.25}\right)$

$= 0.364$. Finally, in Member DE, $R_{de} = \frac{10.0 + 5.5 + 4.8 + 3}{10.0} = 2.33$;

$K_{edm} = 10\left(1 - \frac{0.7 \times 0.7}{2.33}\right) = 7.90$; and $\gamma_{edm} = 0.7\,\frac{1.33}{1.84} = 0.506$.

Consider Members BE, FE, and JE as a group. Determine the values of R_{ed}, K_{dem}, γ_{dem}, etc., and record all these values on the frame (Fig. 8). Evidently, many of these values of R, K_m, and γ_m are not needed.

Step 2.—Only some of the values calculated in Step 1 will be used in Step 2. Assuming an external moment of 100 at Joint D, distribute it among the members meeting at that joint in proportion to their K_{dm} - values (see Fig. 8), as follows:

$$M_{dc} = 4.80 \times \frac{100}{20.79} = 23.08$$
$$M_{da} = 5.50 \times 4.810 = 26.45$$
$$M_{de} = 7.49 \times 4.810 = 36.04$$
$$M_{dh} = 3.00 \times 4.810 = 14.43$$
$$20.79 \qquad\qquad 100.00$$

Carry over the distributed moments by the modified carry-over factors; thus:

$$M_{cd} = M_{dc} \times 0.125 = 23.08 \times 0.125 = 2.88$$
$$M_{ad} = 26.45 \times 0.364 = 9.62$$
$$M_{ed} = 36.04 \times 0.455 = 16.40$$
$$M_{hd} = 14.43 \times 0.500 = 7.21$$

Distribute each moment thus carried over among the connected members at the joint, in proportion to their modified stiffnesses: At Joint C, $M_{cg} = -M_{cd} = -2.88$; at Joints A and H, the frame ends, and no further distribution will be necessary; and at Joint E, the moment, $M_{ed} = 16.40$, will be distributed to the connected members, EB, EF, and EJ, thus:

$$M_{eb} = 1.917 \times \frac{-16.40}{1.917 + 4.56 + 3.0} = -3.31$$
$$M_{ef} = 4.56 \times -1.73 = -7.90$$
$$M_{ej} = 3.00 \times -1.73 = -5.19$$
$$-16.40$$

The foregoing distributed moments will again be carried to the far end of each member, thus, $M_{be} = M_{eb} \times 0.435 = 3.31 \times 0.435 = 1.44$, etc. The resulting moments are shown in Fig. 9.

Example 2.—In Fig. 8, let the fixed-end moments in Member CD be $F_{cd} = -1\,000$; and $F_{dc} = 2\,000$ (see Fig. 10). The unbalanced moment at Joint C is then 1 000; and at Joint D, it is $-2\,000$. Instead of distributing the unbalanced moments separately, a better procedure is as follows: (1) Release all the joints; (2) distribute the unbalanced moment of 1 000 at Joint C to End C of Member CD, or, $1\,000 \times \frac{5.59}{5.59 + 1.50} = 788$; (3) carry over

788 to End D of Member DC by the modified carry-over factor, γ_{cdm}, or, $788 \times 0.390 = 308$; (4) distribute the unbalanced moment of $-2\,000$ at Joint D to End D of Member, DC, or, $-2000 \times \frac{4.80}{4.80 + 5.50 + 7.49 + 3.00} = -462$; and, (5) carry over -462 to End C by the modified carry-over factor, γ_{dem}, or, $-462 \times 0.125 = -58$.

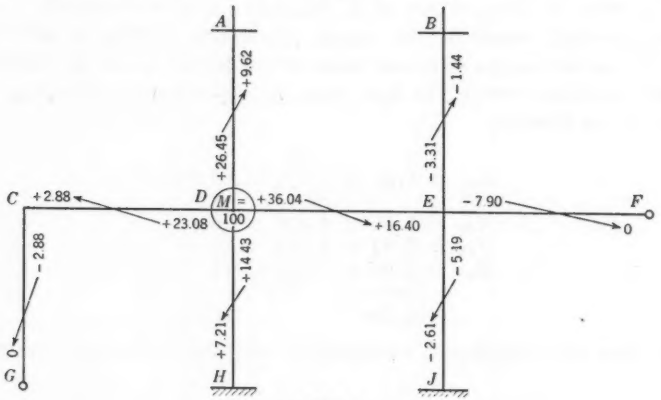

FIG. 9

The total moment at End C of Member CD will then be, $-1\,000 + 788 - 58 = -270$; and the total moment at End D of Member CD will be, $2\,000 - 462 + 308 = 1\,846$. These moments will be entirely resisted by their respective supporting or connected members. They are distributed and carried over to the ends of the frame, as shown in Fig. 10.

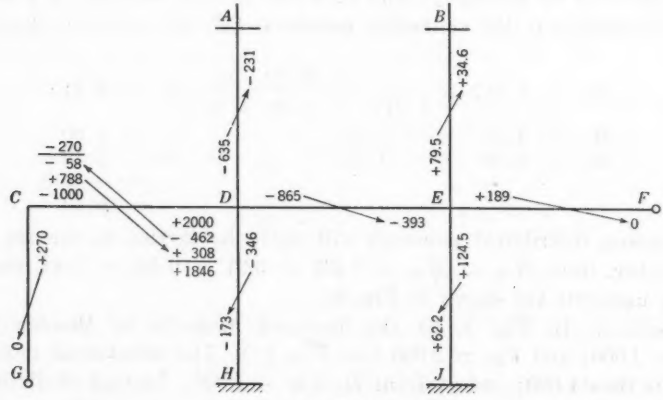

FIG. 10

Example 3.—Consider the frame in Example 1, with fixed-end moments on Girders CD, ED, and EF, as shown in Fig. 11 (identifying symbol, (f)). It is possible to distribute the unbalanced moments at the joints separately,

proceeding as in Example 1, or to find the end moments of each loaded member and proceed as in Example 2. A better method, however, is shown in Fig. 11: (1) Find the unbalanced moment at each joint and distribute it to all the members meeting at that joint, identifying each distributed moment

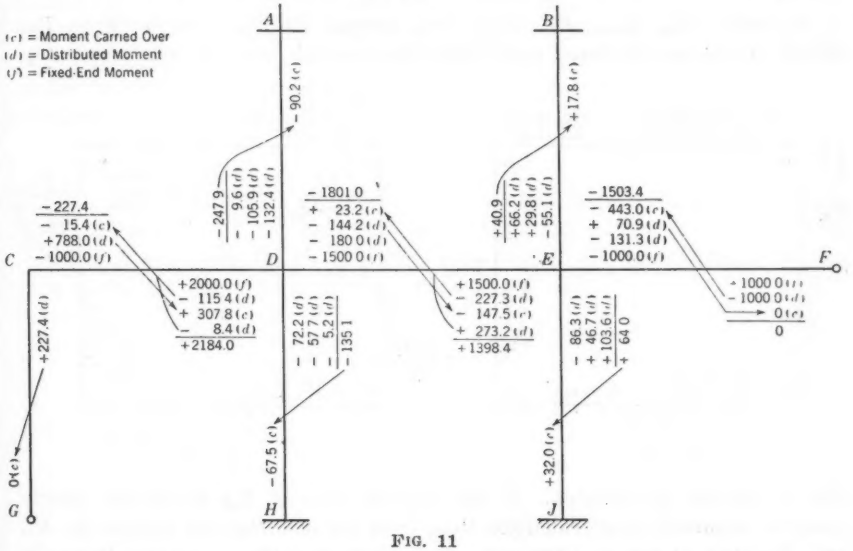

FIG. 11

by the symbol, (d); (2) after distributing the unbalanced moments at all joints, start from one end of the frame, such as Member CD, and carry over the value, $788(d)$, at End C to End D, producing a moment at End D equal to $307.8(c)$, identifying each moment carried over by the symbol, (c); (3) distribute this value, $307.8(c)$, among the supporting members, so that

$$M_{de} = -307.8 \times \frac{7.49}{7.49 + 5.5 + 3.0} = -144.2\ (d),$$ etc.; and (4) combine

the value, $-144.2(d)$, with the first distributed moment, $-180(d)$, at End D of Member DE and carry the sum over to End E, producing a carried-over moment at End E equal to $(-144.2 - 180.0)\ 0.455 = -147.5(c)$. Continue the foregoing steps until all the unbalanced moments are carried to the ends of the frame. Time is saved in this way by combining the moments carried over.

Example 4.—Consider the closed frame shown in Fig. 12. The fixed-end moments are $M_{ab} = -1\,000$; and $M_{ba} = 1\,000$. This box is made up of members of uniform moments of inertia and, therefore, only the K-values need be shown in Fig. 12.

Step 1.—Since the frame is closed, it is necessary, first, to assume the R-value of an end of some member. It can be seen by inspection that

$$R_{ab} = \frac{3 + 1(-)}{3} = 1.3,$$ approximately.

Then, from Equation (3), $K_{bam} = 3\left(1 - \frac{\frac{1}{4}}{1.3}\right) = 2.423$; $R_{bcm} = \frac{1 + 2.423}{1} = 3.423$; and, $K_{cbm} = 1\left(1 - \frac{\frac{1}{4}}{3.423}\right) = 0.927$.

Calculate R_{cd}, K_{dcm}, R_{da}, K_{adm}, etc., around the frame to check on R_{ab}, which, in this case, is found to be 1.302, close enough with the original assumption to warrant no revision. If the checked value of R_{ad} differs too greatly from the assumed value, calculate K_{bam} from the new R_{ab} and change the K_m and R-values as far as necessary. Then, calculate the γ_m-values from the values of R finally adopted.

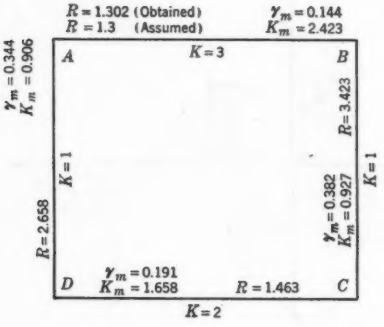

FIG. 12

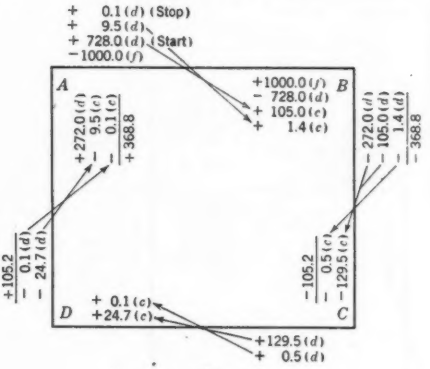

FIG. 13

Step 2.—Distribute the unbalanced moments at Joints A and B (see Fig. 13). Carry over the distributed moment, $728(d)$, at End A of Member AB to End B, producing $105(c)$, which is resisted by $-105(d)$ at End B of Member BC. The moment, $-105(d)$, is combined with the moment, $-272(d)$, at End B of Member BC, and carried over to End C of Member BC. Continue the cycle until the amount to be carried is negligible. For this case of a symmetrical frame and symmetrical loading, if the moment, $-728(d)$, at End B of Member AB is carried over to End A, and the cycle of distribution is performed in the other direction, the moments would evidently be the same, except with reversed signs. Hence, it can be seen that the resulting moments are, $M_{ad} = 272 - 9.5 - 0.1 + 105 + 1.4 = 368.8$; and, $M_{da} = -0.1 - 24.7 + 0.5 + 129.5 = 105.2$.

CONCLUSION

The greatest advantage of this method is its simplicity and directness. Only two formulas need be remembered in addition to the fundamental concepts of moment distribution. Equations (3) and (4) can be derived, and remembered, so easily that an experienced designer can estimate the modified beam factors, mentally, with sufficient accuracy. In these formulas, the effect of fixity of supports is shown by the value, R_{ba}, and the effect of haunching is shown by the value, γ_{ab} γ_{ba}.

It is considered best to limit this paper to a presentation of the method only. Its special adaptability, such as its application to continuous beams of varying sections, has not been discussed. The most interesting application is sometimes found in the approximate but sufficiently accurate estimate of moment by this method. However, it is not recommended as an invariable substitute for the original Cross method. Each has its own advantages, and it is left to the designer to choose for himself. Certainly, it will be worth while for engineers dealing with continuous frames to spend the little time necessary to learn it, so as to be able to utilize its obvious advantages.

Acknowledgments

The writer wishes to express his utmost gratitude to Bruce Jameyson, Assoc. M. Am. Soc. C. E., under whose guidance the study was made. Thanks are also due to John T. Howell, Jun. Am. Soc. C. E., for his valuable suggestions in the preparation of the paper.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

REPORTS

WATER SUPPLY ENGINEERING[1]

PROGRESS REPORT OF THE COMMITTEE OF THE SANITARY ENGINEERING DIVISION

INTRODUCTION

The Committee of the Sanitary Engineering Division on Water Supply Engineering has found no precedent by which to be guided as to the scope of its activities, and considers its report as a preliminary survey which it hopes may lead to a clear-cut policy for the future. The opinion of discussers as to the desired scope would be of assistance.

The Committee is in agreement with views which appear to be held by officials of the Sanitary Engineering Division that any functions that are being effectively exercised by other associations specializing in water supply problems should not be duplicated. It has been suggested that the Committee should attempt to keep itself familiar with important advances in the water supply art and report them to the Sanitary Engineering Division and ultimately to the Society.

PART 1.—ACTIVITIES OF ORGANIZATIONS INTERESTED IN WATER SUPPLY ENGINEERING

Tentatively, the Committee assumed as one of its duties a review of the activities of other committees of the Society, and of other professional societies, whenever these activities touch the water supply field. The aim has been not only to report important contributions by such committees, but also to report any omissions that seem to be of consequence. This survey was begun by communicating with each of the associations (sixteen in number) that appear to have activities touching on water supply engineering. The card catalog of the American Water Works Association was courteously supplied to the Committee for this purpose. The activities of several other

NOTE.—Written discussion on this Progress Report will be transmitted directly to the Chairman of the Committee for possible use in preparing subsequent reports.

[1] Consolidation of First and Second Progress Reports; Part 1 was presented at the meeting of the Sanitary Engineering Division, at New York, N. Y., in January, 1933; Part 2 was presented the following year (January, 1934), and abstracted in *Civil Engineering*, March, 1934, p. 162.

organizations were so well known to the Committee that no communication was required. The complete list of associations considered is, as follows:

American Public Health Association;
American Railway Engineering Association;
American Society for Testing Materials:
American Society of Civil Engineers;
American Society of Mechanical Engineers;
American Society of Sanitary Engineering;
American Standards Association;
American Water Works Association;
Cast Iron Pipe Research Association;
Edison Electric Institute (formerly National Electric Light Association);
Hydraulic Society;
Kansas Water Works Association;
Maryland-Delaware Water and Sewerage Association;
National Fire Protection Association;
National Hydraulic Laboratory, Washington, D. C.;
New England Water Works Association;
Pennsylvania Water Works Association;
Pennsylvania Water Works Operators Association;
South Jersey Association of Water Superintendents;
The Engineering Foundation; and
Virginia Water and Sewage Works Association.

The Secretaries addressed by the Committee were most courteous in supplying all available information.

American Public Health Association

The American Public Health Association numbers among its membership many members of the Society, and of other engineering societies interested in water supply. Among its sections and committees are the following: Laboratory Section; Committee on Lead Poisoning; Committee on Bathing Places; Committee on Plumbing; Committee on Water Supply, through its Sub-Committee on Ground-Water; and, Committee on Water Ways Pollution.

Studies relating to standard methods of analyzing water are assigned to the Laboratory Section. Late in 1932, a sub-committee of this Section, jointly with the American Water Works Association, sent to press a new edition of the Manual on Standard Methods of Water Analysis.

A study of the foregoing committees reveals a considerable proportion of members who also belong to the Society and most of their chairmen are members of the Society.

American Railway Engineering Association

The Secretary of the American Railway Engineering Association writes: "Our functions relate only to the use of water for locomotive boiler use." In the "Outline and Personnel of Committees," published by this Association, Committee XIII (on Water Service and Sanitation), has nine subjects assigned to it, including a revision of the Manual of the Association; pitting of locomotive boilers; methods and economic value of water treatment; deep-

well pumping; methods of analyzing chemicals used in water treatment, etc. Water supply engineers are generally familiar with the active studies of the Water Supply Committee of the Association, which has helped in raising the technique of boiler-water treatment to a high state of practical efficiency and economy in the relatively small plants that serve locomotives especially.

It is noted that one of the most active of the members of the Water Supply Committee of the Association is Chairman of the Committee of the American Water Works Association on Municipal and Industrial Practices in Water Softening.

American Standards Association

The Society and practically all other important technical associations in the United States, as well as departments of the Federal Government, are "member-bodies" in the American Standards Association. This organization provides rules of procedure and supervises the selection of broad representation for committees that are sponsored by one or more member-bodies and are authorized to create a National standard for some material or method. A few of the several hundred A. S. A. projects that pertain more or less directly to water supply are:

Committee A-1, on Specifications for Portland Cement[2];
Committee A-21, on Specifications for Cast Iron Pipe and Special Castings[2];
Committee A-35, on Manhole Frames and Covers[3];
Committee A-36, on Methods of Rating Rivers for Water Power[2];
Committee B-16, on Pipe Flanges and Fittings[4]; and
Committee B-31, on Code for Pressure Piping (that is, design, manufacture, testing, installation, and operation of pressure piping systems).[4]

It may be remarked in passing that the findings of Committee B-31 (in which the Society has no membership) are likely to be incorporated in city and State laws. In 1932, the American Water Works Association secured a modification of the scope of Committee B-31 so as to leave to water supply interests certain matters relating more particularly to water supply piping. This case illustrates the need for watchfulness lest, accidently, some standard that does not provide for the particular needs of water supply engineering should become compulsory. Some group, possibly a committee of the Society, should watch for cases of this kind.

From the beginning Committee A-21, on Specifications for Cast Iron Pipe and Special Castings, has been a research committee as well as a standardization committee, and its researches should be of particular interest to water supply engineers, because they have included experiments in loss of head in fittings; strength of fittings; strength of bells and joints; strength of various kinds of cast-iron pipe under water pressure alone, under earth

[2] The Society has official membership on this Committee.
[3] The Society is one of the sponsors.
[4] The Society has no official membership.

pressure alone, under combined earth loads and water pressure, and under impact; corrosion of different kinds of cast iron; and experiments relating to tar and cement mortar linings for water pipe. Many progress reports have been circulated among the Committee's membership which, with its subcommittees, includes about 100 persons. Committee A-21 has had the advantage of a considerable fund contributed by manufacturers and by the sponsor societies for research work and testing, preliminary to the formulation of specifications. The work has included testing of 12-in., 20-in., and 48-in. pit cast pipes in a great many ways, including combined exterior loads and water pressure. From these tests a rational method of designing pipe barrels has been developed so as to include not only water pressure, but also external loads.

The full series of tests on losses of head in bends, tees, and crosses, described subsequently under the heading, "Loss-of-Head Measurements," have resulted in the conclusion that losses from fittings in distribution systems are, in general, very small, and that, due to the frequency of branches for hydrant connections, there is no advantage in using tees and crosses of wide sweep.

Tests to destruction, of fittings, have indicated unexpectedly low strengths in tees and crosses, and improved designs are being developed. The Committee's specifications for mortar linings for pipe have been issued as proposed tentative standards and are in general use. Other results have not been published for general distribution, but will be available in the not distant future.

American Society of Mechanical Engineers

Water supply engineers and hydraulicians will do well to follow the activities of the American Society of Mechanical Engineers, because its committees are active in some of the branches that pertain to water supply and hydraulic work. Besides being principal or joint sponsor for a number of committees of the American Standards Association, its Hydraulic Division is active in the study of flow in pipes and in the study of water ram. Attention is called to a paper by Mr. Emory Kemler before the December, 1932, Meeting of the A. S. M. E., which appears to systematize formulas for the flow of all kinds of fluids.

Perusal of "Council and Committee Reports for the Year 1931–32" discloses the following activities of interest to water supply engineers:

Fluid Meters Committee: Complete report submitted;
Condenser Tubes Committee: Study of service life of various compositions;
Boiler Feed Water Committee: Joint membership with American Boiler Manufacturers, American Railway Engineering Association, American Society for Testing Materials, American Water Works Association, and Edison Electric Institute: Comprehensive experimental program, begun several years ago, is being continued;
Diesel Fuel Oil Specifications: Committee research work;
Diesel Fuel Consumption: Joint Committee with Society of Automotive Engineers;

Pipe Flanges and Fittings (Sectional Committee B-16 of the American Standards Association): 800-lb Hydraulic Cast Iron Pipe Flanges and Flanged Fittings: Standard completed and published during 1932. Revision of Standard for Steel Pipe Flanges and Flanged Fittings progressing. (In 1932, Committee B-16 completed new flange standards for 50, 125, and 250-lb pressures);

Code for Pressure Piping (Committee B-31 of the American Standards Association): The Editing Committee has received and edited parts of proposed code;

Pipe Threads (Sectional Committee B-2 (1919) of the American Standards Association): Revision of taper thread standards nearly completed;

Wrought Iron and Wrought Steel Pipe and Tubing (Sectional Committee B-36 of the American Standards Association): Tentative report on Pipe and Tubing for Low Service is under discussion;

Screw Threads for Hose Couplings (Sectional Committee B-33 of the American Standards Association): Revision in preparation; and,

Standardization and Unification of Screw Threads (Sectional Committee B-1 of the American Standards Association): Report of Sub-Group on Screw Thread Survey of the United States. Revision of National standards in preparation.

The Society has no official representation in the Boiler Feed Water Committee or in Committees B-16 and B-31 of the American Standards Association. The American Water Works Association is represented on all these committees and at least one member of each committee is a member of the Society.

American Society for Testing Materials

This Association is tremendously active in all branches of specification and research work relating to materials. It is also joint sponsor on several committees of the American Standards Association relating to water supply materials. Not infrequently, A. S. T. M. standard specifications can be quoted by number and title only, in specifications for water supply material. They are internationally known and unnecessary printing in specifications can often be saved by such reference. The most important committees (from the water supply point of view) of the American Standards Association (of which the American Society for Testing Materials is joint sponsor), such as Committee A-21, on Cast Iron Pipes, B-16, on Flanges and Flange Fittings, and B-31 on Code for Pressure Piping, are treated briefly herein under the heading "American Standards Association."

American Society of Sanitary Engineering

The American Society of Sanitary Engineering has a research committee of nine members from various parts of the United States and Canada. Water supply is included in its scope.

American Water Works Association

So many members of the Sanitary Engineering Division of the Society are active members of the American Water Works Association that it is almost unnecessary to publish even a brief list of its manifold activities. For

completeness, however, and for the benefit of those who are not members of the A. W. W. A., a brief summary will be made.

Subsequent to the publication in 1925 of the Water Works Practice Manual, the American Water Works Association organized its technical committee activities with a view to issuing revised editions of the Manual about every five years. A general committee of ten members (all but one of whom are members of the Society) is in charge. Twelve main committees, with about fifty sub-committees, divide the entire field of water-works practice and are engaged upon revised drafts of parts of the Manual.

In addition to the committees engaged on the new edition of the Manual there are the following technical committees: The Committee on Standard Methods of Water Analysis; and the Committee on Boiler Feed Water Studies; Electrolysis and Electrical Interference; Uniform Marking of Fire Hydrants to Indicate Their Relative Fire-Stream Capacity; Licensing Water Works Employees; and Hazards to Plant and Personnel from Use of Chlorine and Other Chemicals in Water Purification Plants. Other joint committees and committees of the American Standards Association in which the American Water Works Association has membership are: A-21 on Specifications for Cast Iron Pipe; A-35, on Manhole Frames and Covers; A-40, on Plumbing Equipment; B-2, on Pipe Threads; B-16, on Flanges and Flange Fittings; B-31, on Code for Pressure Piping; B-36, on Wrought Iron and Wrought Steel Pipe and Tubing; C-1, on Regulations for Electric Wire and Apparatus in Relation to Fire Hazard; G-8, on Zinc Coating for Iron and Steel; and, Z-23, on Sieves for Testing Purposes. A special committee, jointly with a similar committee of the American Public Health Association, has issued the Seventh Edition of Standard Methods of Water Analysis, and the Eighth Edition is in course of preparation.

Inquiry as to policy and progress of the various committees relating to the Manual indicates that the first draft of reports will be long and will be published in the *Journal* of the American Water Works Association, or as separate books, for general discussion before the material is condensed for use in the Manual. Some of the many committees are backward in producing their assignment, but others are making substantial progress. A report on cross-connections was published in the *Journal* in 1933.

It will be noted that A. W. W. A. has not attempted to cover such highly technical subjects as design of gravity and arched masonry dams, so that there is no overlapping with the research committees of the Society and those of the American Society of Mechanical Engineers; neither is there overlapping with any of the committees of the American Standards Association since the A. W. W. A. is closely connected, in general, with those committees either as joint sponsor or through representation in the membership.

Cast Iron Pipe Research Association

The Cast Iron Pipe Research Association is active in research work for improving cast-iron pipe for water supply and other uses. It supplements the work of the American Standards Association Committee A-21, on Specifica-

tions for Cast Iron Pipe. This organization has conducted many bursting tests of water supply fittings and has made improved designs based on these tests. A new general standard for short body fittings is in preparation with the aid of Committee A-21 of the American Standards Association.

Engineering Foundation

The Committee of Engineering Foundation on Arch Dam Investigation, including the Stevenson Creek Arch Dam experiments, has completed its final report. Reports have also been published of important collateral studies on celluloid or mortar models of certain dams, including Stevenson Creek, Gibson, and Boulder Dams.

The Foundation is aiding also the development of a new and promising method of studying stresses, devised at first in connection with investigations inaugurated by the American Institute of Mining and Metallurgical Engineers. The Institute has used the method in problems of mine pillars and props, and other underground problems affecting economy and safety of operations, but it has also been found applicable to dams and other structures. The method utilizes centrifugal force applied to models or to specimens of materials so as to produce any desired degree of stressing. The behavior of the specimen can be recorded photographically whenever desired. Dr. P. B. Bucky, of Columbia University, New York City, has been mainly responsible for this development.

The Hydraulic Society

Judging by the communication from its Secretary, The Hydraulic Society appears to be an association of manufacturers organized for the purpose of considering technical questions in the design of pumps. In 1932, it issued a set of standard curves showing the upper limits of specific speeds for double-suction, single-stake, centrifugal pumps with various suction lifts and total lifts.

Kansas Water Works Association

The Kansas Water Works Association was organized for the purpose of sponsoring a three-day Water-Works Course at the University of Kansas for water-works operators. A *Journal* is published by this Association.

Maryland-Delaware Water and Sewerage Association

The Maryland-Delaware Water and Sewerage Association confines its activities to one annual meeting to promote the interest and education of operators. A *Journal* is published, but the Association has no special or standing committees on research.

Edison Electric Institute

Two of the committees of the Edison Electric Institute (formerly National Electric Light Association) are concerned with water supply engineering: The Prime Movers Committee, which deals with boiler feed water and condenser cooling water; and the Hydraulic Power Committee the activities of

which include forecasting of available sources, the regimen of rivers, and, in some cases, the relation between irrigation and water impounded for power purposes. No research projects have been conducted, except as the Institute is represented on the Joint Committee for Boiler Feed Water Studies under the sponsorship of the American Society of Mechanical Engineers.

Since 1929 the Institute (N. E. L. A.) has published papers on the treatment of feed water (April, 1929, and June, 1930), and three reports on power station chemistry (February and August, 1929, and January, 1931).

National Fire Protection Association

The National Fire Protection Association has for its purpase the promotion of the science and improvement in methods of fire protection and prevention. Its membership includes not only individual architects, engineers, builders, bankers, manufacturers, insurance agents, etc., but other associations, such as boards of trade, chambers of commerce, water-works associations, fire underwriters, etc., to the number of about 150, embracing most of the important associations in any way interested in fire-protection technique. The Society and the American Water Works Association are members.

The publications include, in addition to the *Quarterly Magazine,* numerous standards and manuals, as well as monographs on certain big fires and a variety of subjects relating to fire hazards and protection. Among the standards are those for Fire Pumps, Sprinkler Equipment, and Tanks and Valves. Among the forty or more standing committees are Committees on Fire Pumps, Hydrants, Valves and Pipe Fittings, Public Water Supplies for Private Fire Protection, and Tanks. In each of these committees there are members of the Society, and, in one case, the chairman is a member of the Society.

It is evident that the National Fire Protection Association has activities that parallel some of those of the American Water Works Association and the New England Water Works Association, but nothing that parallels the present activities of the Society.

New England Water Works Association

This is one of the oldest (established in 1882) and most active of regional water works associations and has a considerable membership outside New England. Not a few of its members are also members of the Society.

Its committees cover the following subjects: Reforestation, Meters, Hydrants, Water Fixtures, Pressure and Vacuum Gages, Uniform Statistics, Sanitary Scoring of Water Supplies, Disinfection of Water Mains, Corrosion of Pipes and Standpipes, Soil Corrosion, Tests, Legislation, Flanges and Flange Fittings Standards,[5] Manhole Frames and Covers, Hose Couplings,[5] Code for Pressure Piping,[5] Meter Rates Standards,[5] Wrought Iron and Steel Pipe,[5] Water Works Education, Yield of Catchment Areas, Grounding of Electric Currents on House Plumbing (jointly with American Water Works Association), Standard Specifications, and Thawing Services.

[5] By representation on the Committees of the American Standards Association.

The scope of the committee work is wide, but does not conflict with the present activities of the Society. Where it might otherwise conflict with the American Water Works Association, or other associations, joint committees or sectional committees under the auspices of the American Standards Association are frequently functioning.

Pennsylvania Water Works Association

The Pennsylvania Water Works Association is not a technical association, but one of business men owning private water supplies, and their staffs. Its officers number many members of the Society. Its meetings develop valuable information on financial, legal, valuation, and management problems, some of which is published in an annual volume.

Pennsylvania Water Works Operators Association

The Pennsylvania Water Works Operators Association has about 260 members, including members of the management and operating staffs of various municipal and private water-works, State officials, and some material supply houses. It has no research or standardizing committees at present, but contemplates the creation of some. It publishes a yearly volume containing high-grade papers on various subjects relating to water-works operation.

South Jersey Association of Water Superintendents

The South Jersey Association of Water Superintendents has about forty members, mostly superintendents of public and private water companies in the southern part of New Jersey. It has three technical committees covering the fields of plant operation, purification, and transmission and distribution.

Virginia Water and Sewage Works Association

The Virginia Water and Sewage Works Association was organized in 1929. Its membership is drawn from superintendents and State officials, and its objects include the advancement of knowledge and the adaptation of scientific methods to water and sewage plant operation.

American Society of Civil Engineers

The establishment of Divisions and Sections has increased the number of committees and has tended to call attention to omissions in the activities of the Society. The Committee of the Sanitary Engineering Division on Water Supply Engineering is a product of the recognition, following the establishment of the Division, that water supply in general was not listed as a Society activity.

The Society has the following committees dealing with subjects of special interest to water supply engineers: The Committee on Dams (a Research Committee); the Committee on Meteorological Data (a Research Committee); the Committee of the Irrigation Division on Interstate Water Rights; Joint Committee of the Irrigation and Sanitary Engineering Divisions on Salvage of Sewage; and the Committee of the Sanitary Engineering Division on Water Supply.

An extraordinarily active research committee on Irrigation Hydraulics was appointed June 30, 1922, and published several important progress reports in the succeeding 11½ years. After submitting its final report,[6] the Committee was discontinued, at its own request, on December 31, 1933. Its researches were focused in special papers by members of the Committee. Among the subjects considered, four are of interest to water supply engineers: Evaporation Losses from Reservoirs; Silt; Scouring Below Dams; and Water Movement and Pressure Under Dams.

The Committee prepared a complete bibliography on each subject, a glossary of terms, and a set of standard symbols for writers on these subjects.

Evaporation.—The objective of the study to determine evaporation losses from reservoirs was to derive a set of standard coefficients by which records of evaporation from a large variety of pans could be converted to that from a large water surface. Five papers on this subject have been published. The Committee recommended that for future records two or more pans be used at a location. If only one pan can be used its preference was as follows:[7]

(*a*) Class A land pan, 4 ft in diameter, 10 in. deep, made of No. 22-gauge galvanized iron. Bottom of pan to be placed 6 in. above the ground surface.

(*b*) Modified Colorado "buried pan," 3 ft square, 18 in. deep, made of No. 18 galzanized iron, with no paint. The pan is buried to within 4 in. of its top.

(*c*) United States Geological Survey floating pan. This is of the same size and material as Pan (*b*) and is floated in water with its top 3 in. above the surface.

Pressure Against Dams.—The Committee on Irrigation Hydraulics was partly responsible for the installation of pipes in dams to determine uplift pressures at bed contacts and horizontal joints. Four papers relating to this subject have been published.

The Silt Problem and Scouring Below Dams.—The final report[6] and seven other published papers are available on these subjects.

Suggestions as to Water Supply Activities of the Society

From an examination of committee activities in the various associations one is impressed by the fact that a creditable degree of co-operation has grown up during the decade, 1924–34, and there appears to be little danger of overlapping and conflicting standards. The Society, however, appears not to be so closely in touch as is desirable with committees of the American Standards Association, but has tended to emphasize purely technical research and discussion. Standardization has become so active and so powerful in shaping engineering practice that it would seem necessary for the Society to have strong representation wherever standards affecting civil engineering are under consideration.

The Society has been rather slow in taking the lead in such hydraulic research as would provide water supply engineers and hydraulicians with reliable constants and safe standards for use in practical problems. It seems

[6] *Transactions*, Am. Soc. C. E., Vol. 99 (1934), p. 1375.
[7] *Loc. cit.*, p. 716.

as if the classic experiments of D'Arcy, Bazin, Mills, Fteley and Stearns, and other great hydraulicians of the Nineteenth Century had been used in the Twentieth Century without corresponding contributions to fill gaps with which every designer has still to struggle.

Two research committees of the Society that functioned for many years, (Hydraulics and Flood Flows), ceased to exist several years ago without definite accomplishments. A considerable number of problems which might appropriately have been investigated by the Hydraulics Committee were included in the program of the Committee on Irrigation Hydraulics.

Other problems remain that are of concern to water supply engineers (and other hydraulic engineers) which an active research committee could well undertake. Among these are: (1) Increased losses of head in pipes with various kinds of coatings as affected by age and kind of water; (2) losses of head in fittings, valves, check-valves, and transitions of various common kinds; (3) effect of piers and changes of section on flow in streams, including laws of recovery of velocity head; (4) water-hammer and its control; (5) special methods of measuring water (for example, electrolytes, photopitometry, etc.); and (6) laws of scale relationship of hydraulic models: (*a*) For conduits, channels, etc.; and (*b*) for mixing and sedimentation basins, etc. The first-named subject in relation to steel conduits was considered and a program was laid out by a committee of the American Water Works Association in 1932, but lack of funds has prevented progress.

The systematizing of flood-flow data would be of great use to water supply engineers as affecting the design of spillways and the safety of dams and the population below them, and a way should be found to promote this work. Kindred to it also is the matter of run-off records. Without earnest pressure from water supply, irrigation, and canal engineers, records of many important streams will doubtless be interrupted. The Society, and perhaps through it, American Engineering Council, may aid in promoting Government stream-gauging work.

The matter of gravity masonry dams is also one on which there has always been need of more authoritative pronouncement, since it is still not clear what upward water pressure and what ice pressure should be allowed and what factor of safety should be demanded. Two strong committees of the Society are studying this subject, but dams continue to be designed with greatly varying assumptions.

Flood flow and spillway provisions continue also to be widely at variance for lack of adequate authoritative pronouncements. Water-hammer and its possibilities as to piling up destructive pressures by coincidence of two or more wave peaks is still an enigma which may be responsible for unexplained failures. It should be clarified.

Many competent engineers are unemployed. Examples of useful employment of such engineers on technical investigations paid for with relief funds have been furnished in certain localities, notably Massachusetts. The Committee suggests the possibility of similar work on some of the hydraulic research projects mentioned in this report, notably gaugings of pipe lines.

Notable Advances

The Committee has been considering what it should do as to reporting yearly to the Division, advances in the art of water supply engineering. At first, this may seem an axiomatic and simple proposal, but much thought was given to it by a previous committee. The only feasible method appeared to be by means of a considerable enlargement of committee personnel, possibly by appointing affiliates and thus covering the field which is so extensive and advancing so rapidly that without such help the results would be provincial and localized—not worthy of the national scope of the Society.

The collection of data or raw material might be fairly simple, but meetings could not be held without funds, the appropriation of which is almost out of the question, at least for the immediate future.

As examples of the kind of advances which might be reported, the Committee has listed the following: (*a*) Improvement in methods of developing deep graveled wells so as to increase the yield greatly; (*b*) the development of durable bituminous and cement mortar coatings for the interior of cast-iron and steel pipes, leading to a greatly increased life expectation of little-impaired carrying capacity; (*c*) the development of clearer principles governing allocation of interstate waters; (*d*) improvement in the art and technique of designing hydraulic and semi-hydraulic fill dams; and (*e*) the application of activated carbon, of ammonium chloride, and of fullers' earth in removing tastes and odors from water. The mere listing would not be of much assistance. Brief but adequate description, reference to sources of information, and finally, perhaps, appraisal involving personal opinions of the members of the Committee. The latter is a question requiring careful consideration, and it is recommended that the Committee give it further thought with the help of the general officers of the Sanitary Engineering Division.

As to the question of hydraulic research, the Committee is tentatively of the opinion (subject to discussion with the officers of the Division and the Society) that this falls within the scope of the Society's research committees and that the Committee of the Sanitary Engineering Division on Water Supply Engineering can best function by making suggestions. An exception might be made, perhaps, in obtaining coefficients of existing pipe lines which does not involve theory so much as knowledge of the location of favorable opportunities and the organization and instruction of parties competent to make gaugings. Aid from a research committee as to methods might well be sought in this case.

PART 2.—STATUS OF THE ART OF WATER SUPPLY ENGINEERING

Introduction

A review of progress made by the Committee to date is incorporated in Part 2. In this work help has been sought from many non-members and sincere acknowledgment is made for their assistance.

At the beginning of its labors the Committee outlined a series of topics which was distributed among the members of the Committee and thirteen non-members in various parts of the country. The widest latitude was invited as to the inclusion of other topics under which advances could be indicated. These topics were, as follows:

1.—Examples of new types of construction (fabrication, design, or processes), involving the application of principles or methods not previously used, or at least used only in a few cases not well known. This might cover improvements in:

(*a*) Placing earth-fill in dams, of controlling the selection of material used, or of measuring the degree of compactness obtained.
(*b*) Placing concrete.
(*c*) Cooling concrete in the heart of thick dams.
(*d*) The prevention of corrosion of water pipes with resulting coefficients of flow where available.
(*e*) Applications of chemicals used in the purification of water.
(*f*) Arrangement of baffles, or other devices for facilitating sedimentation.
(*g*) Purification processes, or novel arrangements of common purification devices.
(*h*) Pump, motor, or engine designs.
(*i*) Check valves.
(*j*) Types of pipe.

2.—Examples of advance in the more theoretical or experimental aspects of water supply engineering, such as:

(*a*) Water-hammer and its prevention.
(*b*) Measuring devices, such as Venturi flumes.
(*c*) Loss-of-head measurements useful in the design of piping.

3.—New materials developed or adapted to water supply engineering, such as:

(*a*) Cement of special chemical composition.
(*b*) Application of new alloys, bronzes, etc., of improved strength or corrosion resistance.

4.—Progress of such committees as: The Committee of the Society on Dams; the Committee of the American Standards Association, on Cast Iron Pipes; and various committees on steel pipes.

Not all the foregoing topics were covered in the various replies, but the Committee, in assembling the data, has filled in all possible gaps. Part 2 is arranged somewhat in accordance with this list of topics, but with some changes in order.

Cement

In recent years the significant development is a trend away from the so-called "standard specifications" for Portland cement. The advanced engineer is now specifying cement to meet his particular needs. Quick-setting cement is necessary for highways, whereas slow-setting and low-heat cement is required for massive masonry structures.

Rapid hardening of cement is accompanied by great evolution of heat, and it is possible that this desirable quality is obtained at the expense of later durability and permanence, particularly in exposed locations.

The rapid-hardening variety depends mainly on an increase in the lime content which is thought to cause an increase in the more active constituent, tri-calcium silicate. The slow-setting, low-heat cements are made with less lime, less alumina, more ferric oxide, and more silica. They harden with relative slowness and give rise to much less temperature than occurs with cement of standard specifications. In general, these cements give promise of maximum durability under severe conditions of exposure and show a maximum resistance to the action of "alkalis", such as sodium sulfate and sea water.

Advances in Construction, Fabrication, Design, Etc.

Placing Concrete.—Methods unusual in masonry dams were used in Pine Canyon Dam. About 500 000 bbl of cement were purchased under specifications requiring that the heat development during setting should not be more than 65 calories per gram at 7 days, nor more than 80 calories per gram in 28 days. Actual tests of the cement used showed results of 55 and 65 calories, respectively.

The concrete in the body of the dam had a cement-water ratio of 1.68 and a slump of only 1 in., or less. Internal and external vibrators were used, the internal type giving the most satisfactory concrete. The concrete averaged 0.95 bbl of cement per cu yd and weighed 156 lb per cu ft. Test cylinders, 14 in. in diameter by 28 in. high, developed crushing strengths of 2 000 lb per sq in. at 28 days, and 3 000 lb at 90 days. These results are about 20% less than those obtained by the more usual test cylinders having a diameter of 6 in. and a height of 12 in., with all aggregates larger than $1\frac{1}{2}$ in. screened out.

Cooling Concrete at Boulder Dam.—The chemical reaction of setting concrete releases heat estimated to average 40° F above the temperature at which it is poured. The time required for a concrete structure to lose a given proportion of excess heat, other things being equal, is proportional to the square of the thickness.

The purpose of cooling the concrete at Boulder Dam is to complete the shrinkage before the dam is made into a structural unit by grouting the joints. Excess heat is being removed in a short period by circulating cold water through 1-in. steel pipes which are embedded in the concrete as it is placed. The pipes are spaced about 5 ft 9 in. from center to center throughout the mass. Approximately, 1 ft of embedded pipe is required for each cubic yard of concrete to be cooled.

Equipment for cooling water consists of a 3-unit refrigerating plant having a total capacity of 600 tons of ice per day, and a cooling tower of a rated capacity of 6 000 gpm, to which the heated water is returned for re-cooling. More than one-half the heat of the concrete will be extracted by water pumped directly from the cooling tower. The final process is supplied by the refrigerating plant.

Earth Dams.—No new principles have been applied recently in the construction of earth dams, but doubtless there is a more general understanding of the selection of material by determining its perviousness, moisture content, and grain size in hydraulic-fill dams, as illustrated by the Cobble Mountain Dam, of the water supply for Springfield, Mass. In compacting rolled dams the "sheep's-foot" roller has been used increasingly, particularly in the West. This has been found to give a high concentration of pressure on the material to be rolled. An almost equally high degree of pressure was also obtained in the Chenery Reservoir, of the Contra Costa Water Company, near San Francisco, Calif., by the use, for transporting material for the dam, of 7½-ton, solid-tire, trucks each loaded with about 7 or 8 cu yd of material.

The unit weight of rolled embankment is frequently checked, either by weighing cores cut by a special steel tube, or by weighing material cut from measured excavations of approximately 1 cu ft.

Pipe and Pipe Coatings

Fabricated Steel Plate Pipe.—The general acceptance of automatic fusion welding for shop fabrication has made possible a steel plate pipe with a smooth waterway, the field joints being fabricated either by fusion welding or by couplings of a special type, or by both, in the same line. An Eastern example is the 10-mile line of 30-in. steel pipe of the Panther Valley Water Company, at Tamaqua, Pa., laid in 1932. In this pipe four 30-ft lengths were assembled over the trench and the field joints butt-welded with a metallic arc, the 120-ft assemblies being then connected in the trench by couplings. A Western example is the 5½-mile, 78-in. line laid in Seattle, Wash., in 1933. In this line the field joints are double-lap welds, one end of each pipe being expanded to form a bell into which the end of the adjacent pipe fits with a lap of about 2½ in., the joint being then welded inside and out.

Cast-Iron Pipe.—The trend in the field of cast-iron pipe is definitely toward the centrifugal type. While production during 1933 was at a low ebb at least two important foundries closed their pit cast shops and are concentrating on centrifugal pipe.

Spiral Welded Pipe.—Steel pipe made with plates wound spirally and with joints both interlocked and electrically welded, has shown good qualities under bursting tests in the laboratory, and has come into the market to a limited extent in competition with steel pipes made with the usual longitudinal joints. It is regularly supplied with spun mortar lining and exterior coatings or wrappings as required by the customer.

Pipe made of asbestos fiber and Portland cement, consolidated under heavy pressure, is a new entry in the American pipe field although it has been used extensively for some years in Italy. It is offered for resistance to tuberculation by water, to soil corrosion, and to electrolysis, and, in industrial plants, to corrosive gases. In addition to industrial installations, experimental municipal water supply installations of a few hundred to a few thousand feet have been made since 1932, to combat soil corrosion, tuberculation, electrolysis, or red water trouble.

Protective Coatings and Linings

Cement Mortar Linings.—Cement mortar linings for cast-iron pipe have become widely accepted. For several years the City of New York has overcome one of the objections sometimes found to cement-lined pipe—that is, hard and caustic water when the pipe is first put into service—by having all its cement linings coated at the shop with an asphalt paint selected so as not to impart taste or odor to the water. Flow tests of a 16-in. cement-lined pipe at Charleston, S. C., indicated a coefficient of 135 in the Williams and Hazen formula in 1923 when the pipe was new and practically the same (actually 143) in 1933. Several tests on Long Island show coefficients of about 140 in cement-lined pipe two or three years old.

Cement mortar linings and coatings for large steel pipe are coming into use. In California a number of lines up to 60 in. in diameter have been lined and coated at the shop, the lining being applied centrifugally and, in some cases, a reinforced envelope of mortar or gunite is then placed on the outside. To provide stiffness for rail and truck transportation and laying, it has been the general practice in the larger sizes of pipe to make the combined thickness of lining and jacket about 2 in. The water supply tunnel under the entrance to Vancouver Harbor, completed in 1933 for the Greater Vancouver Water District, is lined for water-tightness with thin steel pipe having a 2-in. lining of fine concrete centrifugally applied at the shop and delivered to the site by barge. These pipes, when lined, were 7.50 ft in diameter for the tunnel and 8 ft in diameter for the shafts; the shells were only 5/32 in. thick and to diminish the weight in handling no outside mortar was placed at the shop, the space between the pipes and the tunnels being filled solidly with concrete. A machine for applying a smooth mortar lining to a 36-in. pipe, or larger pipe in place in the ground, has been demonstrated successfully on a short length.

Cement Wrapping.—Cement wrapping has been applied by machine as an outside protection on two important lines. Twenty-five miles of 56-in. and 66-in. fusion-welded, steel pipe with riveted field joints in 30-ft lengths was wrapped in 1930 and 1931, with a ¾-in. mortar envelope for the Hetch Hetchy Aqueduct in California, and 5½ miles of 78-in. fusion-welded steel pipe with double-welded lap field joints was wrapped with ½ in. of mortar in 1933 for the City of Seattle. The pipes are revolved in a machine and a ribbon of mortar, supported on a ribbon of wire-mesh reinforcement, is wrapepd around the revolving pipe by a traveling carriage. The tension on the wire mesh forces it into the mortar and the entire unit is finished and compacted by a ribbon of cotton fabric wound on the outside under suitable tension. Mechanical vibration is also used to compact the mortar. The field joints in the mortar envelope are made by hand-wrapping with mortar, wire mesh, and cotton fabric.

Hard Bituminous Enamels.—Hard, bitumastic enamel, centrifugally applied, has been used for several years in cast-iron pipe. The hard coal-tar enamels applied at the shop have sometimes caused difficulty when exposed

to a wide range of winter and summer temperatures in the purchaser's yard. During 1933 small centrifugal machines have been developed, with which the pipe purchaser can apply bitumastic enamel to tar-dipped pipes in his own yard immediately before laying. The Washington (D.C.) Suburban Sanitary District, the New Haven (Conn.) Water Company, and the Water Departments of Newark, N. J., and Newton, Mass., have been lining their own pipes by this method.

Testing Bituminous Coatings.—During the past year or more a method of testing for pin-holes and "holidays" in heavy bituminous coatings has been under development, making use of a high-potential electric spark. This method has been applied to such coatings on the outside of a number of gas and oil lines. It appears to be capable of discovering every pin-hole, large air-bubble, or other important mechanical defect in the coating and in its use; each defect discovered is immediately repaired, and the repaired pipes are re-tested. Provided the pipe can be laid and back-filled without damaging the coating, it promises to make possible coatings substantially 100% mechanically perfect. The method appears to be entirely applicable also, to the inspection of heavy bituminous coatings on the inside of large water pipes and penstocks.

A review of the recent progress in pipe coatings would be incomplete without reference to the admirable co-operative investigation of underground corrosion and pipe coatings being conducted by the National Bureau of Standards, the American Gas Association, the American Petroleum Institute, and many individual utilities and manufacturers of pipe materials and coatings. This investigation is rapidly placing the entire difficult problem of underground corrosion and protective coatings, and the economics of the protection of buried pipe lines, on a sound engineering basis.

Taste and Odor Removal

Activated carbon will remove any objectionable taste or odor likely to be present in a water supply. It is effective when all other methods fail. There is a great difference in absorptive capacity of various carbons, and a very active carbon is more than one hundred times as effective as ordinary charcoal. Activated carbon is effective and economical for the removal of excess chlorine. It is not necessary, however, to use an activated material for this purpose. For such tastes and odors as may be destroyed by an excess of chlorine any material which dechlorinates the water is satisfactory.

Fuller's earth for removing tastes and odors has not proved its value, although several operators report their belief that its use is worth the cost. One operator reported to the Committee his belief that the main advantage of Fuller's earth is in inducing an extra heavy alum floc which, in turn, aids in the removal of tastes and odors.

Some other materials containing carbon or potassium permanganate have been used probably with some effect, but not equal to the effect of activated carbon.

Application of Chlorine in Water Purification

Improvements continue in the methods of applying chlorine both with and without ammonia, and by determining the quantities which are required. It may almost be said that a new technique has been developed for the determination of chlorine residuals, and the water-works operator is advised to consult recent literature on this subject if he has not kept himself abreast of the progress in this respect.

The usefulness of pre-chlorination in heavy doses and areation for the treatment of badly polluted supplies and those having tastes and odors has been proved in recent installations.

Baffles for Sedimentation Basins

F. P. Larmon, M. Am. Soc. C. E., has applied the slatted cross-baffle successfully in several sedimentation basins which were not giving full satisfaction. Such baffles have been common in Imhoff tanks, but the Committee is not aware of their previous application for water purification.

Sealing of Wells Against Salt

The problem of sealing wells against salt, a troublesome one in some well fields, where both salt-water and fresh-water strata occur, with clay separation, has been solved, according to J. A. Wade, M. Am. Soc. C. E., in a well driven at San Mateo, Calif., by the use of two casings with an annular space about 2 in. wide between. This space was filled by oil-well grout poured in through a pipe, the lower end of which was kept below the surface of the rising grout. The narrow space outside the outer casing, left by the steel shoe, is to be filled with clay.

Improved Pumps and Prime Movers

The Committee has canvassed the field of pumps and prime movers with considerable care. As is well known to water supply engineers the centrifugal pump has replaced all other forms to a large extent. One manufacturer reports his increase in marketing of centrifugal pumps, as follows:

Year	Total capacity sold, in million gallons daily
1907	25
1910	150
1915	1 350
1920	3 500
1930	14 000

The increased use of the centrifugal pump is due to higher efficiencies, better adaptation through a wider range of conditions, the possibility of a single-stage unit against higher head, proof of low maintenance cost, and ability to maintain approximately the original efficiency.

Whereas, twenty years ago, efficiencies of 80% were high, efficiencies up to 91.5% are now obtainable in the larger sizes. In April, 1933, an efficiency of 91.5% was offered in Chicago, Ill., in a contract for six pumps (50-mgd,

514 rpm, against a head of 136 ft). The increase in efficiency has been due primarily to extensive shop testing and experimental development under conditions of keen competition among the manufacturers. The improvement in efficiency has been accomplished primarily by better hydraulic design of impellers and casings, reduction in cavitation, and higher speeds available from the driving element. No improvements can be attributed to mechanical features, such as close clearances, which are difficult to maintain and, therefore, such improvements have been due to designs which maintain their value and are not easily lost by wear.

Commenting on the disappearance from the field of the high-duty triple-expansion pumping engine which obtained a maximum duty of 215 000 000 ft-lb per 1 000 lb of steam, one contributor gave a notable example of the frequently observed reduction in space required when substituting centrifugal for pumping engine units. A 60-mgd, steam-turbine, gear-driven, centrifugal, pumping unit operating against a 300-ft head was placed in the same floor space as was previously required by a 20-mgd, vertical, triple-expansion, pumping engine. The duty of the new unit was greater than 195 000 000 ft-lb per 1 000 lb of steam, as compared with approximately 214 000 000 ft-lb for the pumping engine. If more modern high-pressure and high-temperature steam had been available, the turbine-driven unit could have been made to equal the pumping engine in duty.

The electric motor is being increasingly used as a prime mover, because of its convenience, the little attendance required, its high efficiency, and the greater availability of electric power. Synchronous motors for installations larger than 100 hp are generally used, and efficiencies have been generally increased. The first cost is practically the same as that of the induction motor. At Chicago, in the installation previously cited, one bidder named an efficiency of 98.41% for synchronous motors of 1 500 hp. Tests of synchronous motors recently installed in New Jersey and having 350 to 650 hp averaged 96.5% at full load.

There is an increased use of the Diesel engine in sizes of less than 400 hp. This is the result of a growing recognition of the economy and reliability of this type of prime mover, emphasized in some municipalities, by the high cost of electric energy, and, in others, by the continued low cost of fuel oil.

That there is a continuing field for steam turbines in pumping plants is shown by the following information abstracted from an interesting monograph written for the Committee by A. P. Pigman, Chief of the Mechanical Department of the American Water Works and Electric Company. Mr. Pigman regards the high-speed turbine as the outstanding development in steam pumping-station practice during the past few years. Turbines with high thermal efficiency and low first cost are now available in capacities as low as from 200 to 300 hp. The result is that these efficient drives are now installed almost exclusively in steam plants requiring more than 200 hp. When the high turbine efficiencies are united with a steam cycle using high steam pressure, super-heat, and high condenser vacuum, it is not unusual

to obtain a water rate of about 11 to 12 lb of steam per bhp-hr on smaller turbines, 9 to 10 lb on turbines of 1 000 hp capacity, and 8 to 8½ lb in some of the larger water-works turbines ranging in size from 2 500 to 5 000 hp.

In the larger sizes these turbines generally run at a speed of 3 600 rpm, and the speed increases, with the smaller sizes, to 7 000 or 8 000 rpm. The number of stages generally varies between five and fifteen, the larger, lower-speed turbines having the greater number. With careful engineering a turbine and condenser, including speed-reducing gear for driving the pump, can be installed complete, with foundations and auxiliaries, at a cost of between $40 and $60 per hp over a range of sizes from 300 to 5 000 hp, and somewhat more than that for sizes between 200 and 300 hp. These turbine installations lend themselves to control, partly automatic and partly by hand, by which both pressure and quantity of water pumped may be varied as required by consumption or, in some cases, by extra pressure needed in case of fire.

The Committee is unable to report much progress in economical speed control of alternating-current electric motors. Double-wound motors (which are practically two motors in one housing and have two synchronous speeds such as 720 and 514 rpm, or 1 200 and 900 rpm), are sometimes used for sewage plants. Occasionally, four speeds are provided by using, in addition to the double-winding, a pole-changing switch which connects the poles either in series or by pairs in parallel, thus providing four speeds, such as 1 200, 900, 600, and 450 rpm. These changes in speed are too great to provide for flexibility in a water pump supplying variable demands. A brush-shifting type of motor which permits practically any speed from 100% to 33% is often used for printing presses and similar service, but its cost is practically triple that of the synchronous motor and quadruple that of the squirrel-cage motor.

Some years ago a device for speed-control was developed in Switzerland, which also permits practically any speed which would be required. The regulation is accomplished electrically at slight loss in efficiency. The device is economically applicable to motors of larger size, say, 400 hp. and up. Unfortunately, the system appears to be economically applicable only to a constant torque load, such as that required to drive a triplex pump against a constant head. Apparently, this device has not made headway in America. In smaller sizes, mechanical speed-adjusting gearing has been developed to some extent.

Speed control is a field that requires further development in order to permit something like the degree of flexibility for the centrifugal pump that has made the high-duty pumping engine so convenient in the operation of water-supply plants.

Improved Check Valves

The ordinary swing check valve, while cheap in first cost, is expensive to operate, using generally not less than 2½ velocity heads. One valve manufacturer has been conducting many experiments and has made material

improvements in the water way of swing checks, reducing the friction loss to less than one-half its former value. The cone type of check valve (which, when open, gives a full round port) and other check valves of the needle type, have been increasingly used both to reduce friction head and in connection with the control of water-hammer.

Hydraulics

Water-Hammer.—The Society has not been active in the study of the important subject of water-hammer. The American Society of Mechanical Engineers has inaugurated important studies on this problem and has invited co-operation from other agencies.

This subject enters directly into the design of pipes, pumps, and water containers of all kinds. Standardization committees having to do with pipes and fittings have been considerably embarrassed by lack of authentic information.

Venturi Flume

A 10-ft Parshall measuring flume has been installed at Providence, R. I., utilizing the principles of a Venturi meter and attaining similar accuracy. This device is beginning to find use in measuring water or sewage in the hydraulic grade line.

Loss-of-Head Measurements

An extended series of loss-of-head measurements on 6-in. bends, tees, and crosses was completed in 1930 for the American Standards Association's Sectional Committee A–21, on Specifications for Cast Iron Pipe and Special Castings. These experiments are of particular interest because the tees and crosses were of four designs, having sharp corners, 1-in. radii, 2½-in. radii, and 6-in radii, respectively, the latter corresponding with standards of the American Water Works Association. These data can be consulted at the office of the American Water Works Association, in New York City.

Tendencies in Regulation of Utilities by Public Service Commissions

The economic "depression" has accelerated the re-study of methods of utility accounting, valuation, going concern value, depreciation, and rate-making. There is a tendency to change methods (including those of private water companies), which have been more or less generally used in recent years.

The results of any deliberations by the Committee of the Engineering Economics and Finance Division of the Society, on Valuation Procedure and Depreciation, extending the work of the Special Committee of the Society to Formulate Principles and Methods for the Valuation of Railroad Property and Other Public Utilities,[8] will be timely and of great interest to members of the Sanitary Engineering Division engaged in water supply work.

Acknowledgments

In addition to the sources cited directly in the text, the Committee wishes to acknowledge its indebtedness to the following for supplying information

[8] *Transactions*, Am. Soc. C. E., Vol. LXXXI (1917), p. 1311.

essential to its report: John R. Baylis, S. O. Harper, Thaddeus Merriman, S. B. Morris, and A. V. Ruggles, Members, Am. Soc. C. E., with Alex Maxwell, Director, Edison Electric Institute; L. P. Wood, Designing Engineer, New York City Board of Water Supply; De Laval Company; Worthington Pump Company; Allis-Chalmers Company; Fairbanks-Morse Company; American Water Works and Electric Company; and F. R. Berry, Chief Engineer, American Water Works and Electric Company.

Respectively submitted,

THOMAS H. WIGGIN, *Chairman*,
F. A. BARBOUR,
L. R. HOWSON,
C. G. HYDE,
F. H. STEPHENSON,

October 3, 1934

Committee of Sanitary Engineering Division on Water Supply Engineering.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

DISCHARGE FORMULA AND TABLES FOR SHARP-CRESTED SUPPRESSED WEIRS

Discussion

BY C. G. CLINE, ESQ.

C. G. CLINE,[13] ESQ. (by letter).[13a]—As pointed out by Mr. McMillan, the results of the Schoder and Turner experiments must be presumed to be correct until they are proved otherwise. The fact that they do not agree with many of the standard formulas must be taken as an indication that the formulas are at fault.

Professor Draffin discusses a number of formulas and shows how they compare with the Schoder and Turner experiments. Messrs. McMillan and Draffin both agree with the writer that a standard design of suppressed weir should be adopted, a set of carefully planned experiments should be made, and a standard discharge formula and tables computed to agree with the results of the experiments. The computations can be made by using the method of least squares to determine the best numerical values of the coefficients in the algebraic formula selected. This general procedure can be followed without necessarily retaining the particular type of formula used by the writer as an illustration. Mr. McMillan has shown that there need be no scarcity of algebraic formulas; his Equation (35) is relatively simple and yet seems to be sufficiently flexible to cover the entire range of head and height of weir.

In the meantime, as Mr. McMillan states, either his Equation (35), or the writer's Equation (18) and the tables that have been computed from it, may be applied with confidence for general use.

NOTE.—The paper by C. G. Cline, Esq., was published in January, 1934, *Proceedings*. Discussion on this paper appeared in *Proceedings* as follows: May, 1934, by Jasper O. Draffin, M. Am. Soc. C. E.; and September, 1934, by W. Bruce McMillan, M. Am. Soc. C. E.

[13] Senior Asst. Engr. Water Power and Hydrometric Bureau, Dept. of the Interior, Canada, Niagara Falls, Ont., Canada.

[13a] Received by the Secretary October 26, 1934.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

ANALYSIS OF SHEET-PILE BULKHEADS

Discussion

BY THEODOR VON KÁRMÁN, M. AM. SOC. C. E.

THEODOR VON KÁRMÁN,[32] M. AM. SOC. C. E. (by letter).[32a]—The analysis of sheet-pile bulkheads in this paper is a comprehensive presentation of the problem of designing sheet-piles loaded with active or passive earth pressure. In some respects this analysis constitutes a desirable extension of the known methods; for example, (*a*) Mr. Baumann has, fortunately, hit upon the idea of introducing reliable values for the passive earth pressure which has been measured experimentally; and (*b*) he takes into account the influence, on the load distribution, of the yielding of the soil under the passive earth pressure. After formulating the problem in the proper way, Mr. Baumann develops a method of calculation analogous to that used in the case of elastically supported beams, applying the theorem of least work. This method represents progress.

Attention is called to Equation (63), in which it appears that the term corresponding to the work stored in the soil by the passive earth pressure, is incorrect. Mr. Baumann calculates the yield, y_o, due to unit load and makes the work equal to $\frac{1}{2}\, y_o\, p^2$, in which, p is the actual value of the load. The application of this procedure seems to be restricted to the case of a linear relation between yield and load. However, in Mr. Baumann's analysis, the load, p, is connected with the yield, y, by a non-linear relation, $p = C\sqrt{y}$, in which, C is a given function of the depth $(z - h)$. In this case, the term to be varied should be written, $\int_o^p y\, dp = \int_o^p \frac{p^2}{C^2}\, dp = \frac{1}{3}\,\frac{p^3}{C^2}$, whereas Mr. Baumann writes, $y_o = \frac{1}{C^2}$, and uses the expression, $\frac{1}{2}\, y_o\, p^2 = \frac{1}{2}\,\frac{p^2}{C^2}$, for the work, A.

NOTE.—The paper by Paul Baumann, M. Am. Soc. C. E., was published in March, 1934, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: In May, 1934, by Jacob Feld, M. Am. Soc. C. E.; August, 1934, by Messrs. R. L. Vaughn, M. A. Drucker, and Raymond P. Pennoyer; October, 1934, by D. P. Krynine, M. Am. Soc. C. E.; and November, 1934, by Dr. Ing. E. h. O. Franzius.

[32] Prof. and Director, Aeronautical Laboratory, California Inst. of Technology, Pasadena, Calif.

[32a] Received by the Secretary November 5, 1934.

It might be useful to explain the difference between the two viewpoints by a simple example. Assume that a beam is supported by two fixed supports at its two ends and has a "yielding" support in the center (see Fig. 22). The law between the reaction force, Y, and the "yield," y (that is, the deflection

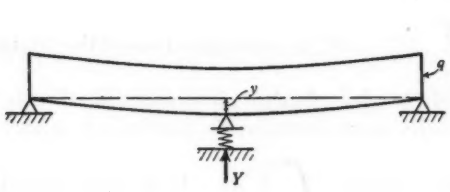

FIG. 22

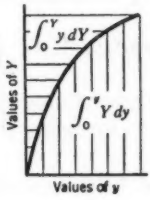

FIG. 23.

of the beam), is given by the general relation, $y = f(Y)$. The beam is loaded by the uniform load, q; denoting the span by l, the deflection in the center, without taking into account the yielding support, would be:

$$y_1 = \frac{5}{384}\,\frac{q\,l^4}{I\,E} \qquad (97)$$

The deflection corresponding to the force, Y, alone would be:

$$y_2 = -\frac{1}{48}\,\frac{Y\,l^3}{I\,E} \qquad (98)$$

Hence, the total deflection, $y = y_1 + y_2$, is equal to:

$$y = \frac{5}{384}\,\frac{q\,l^4}{I\,E} - \frac{1}{48}\,\frac{Y\,l^3}{I\,E} \qquad (99)$$

and the value of the reaction, Y, is given by the equation:

$$\frac{5}{384}\,\frac{q\,l^4}{I\,E} - \frac{1}{48}\,\frac{Y\,l^3}{I\,E} = f(Y) \qquad (100)$$

The same problem can be solved by the theorem of least work. The elastic work stored in the beam is:

$$A_1 = \tfrac{1}{2}\int_0^l \frac{M^2}{I\,E}\,dx \qquad (101)$$

in which, M is the total moment due to the load, p, and the statically unknown reaction, Y. It is known (or, it can be shown by simple calculation) that $-\dfrac{dA_1}{dY}$ is equal to the deflection, y, at the point of application of the statically unknown force, Y. Therefore, Equation (100) which determines Y, can be written in the form:

$$-\frac{dA_1}{dY} = f(Y) \qquad (102)$$

Making $\int_0^Y f(Y)\, dY = A_2$, Equation (102) is equivalent to the statement:

$$\frac{d}{dY}(A_1 + A_2) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (103)$$

or, $A_1 + A_2 =$ minimum.

The quantity, $A_2 = \int_0^Y f(Y)\, dY$, is sometimes termed the "complementary work" ("Ergaenzungsarbeit"). Its physical interpretation is obvious from Fig. 23. It represents the difference between the product, $Y\ y$, and the actual work stored in the yielding support, $\int_0^y Y\, dy$. It is easily seen that in the case of the linear law:

$$\int_0^Y y\, dY = \int_0^y Y\, dy = \tfrac{1}{2}\, y\ Y \quad \ldots\ldots\ldots\ldots\ldots\ldots (104)$$

Hence, in this case, the value of the reaction, Y, is determined by the minimum of the total work; that is, of the sum of the work stored in the beam and that stored in the elastic support.

It is obvious from this example that Mr. Baumann should introduce the term denoting complementary work in his Equation (63). However, it is believed that this minor inconsistency does not materially detract from the value of his new analysis of the sheet-pile bulkhead problem.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

SAND MIXTURES AND SAND MOVEMENT IN FLUVIAL MODELS

Discussion

BY JOSEPH B. TIFFANY, JR., JUN. AM. SOC. C. E., AND CARL E. BENTZEL, ESQ.

JOSEPH B. TIFFANY, JR.,[52] JUN. AM. SOC. C. E., and CARL E. BENTZEL,[53] ESQ. (by letter)[53a].—The results of Captain Kramer's work in Germany have been of especial value to the Staff at the U. S. Waterways Experiment Station, at Vicksburg, Miss., where they have been used as the starting point for a series of investigations of bed-load movement. Nine sand mixtures were tested in a tilting flume, ranging in mean grain size from 0.205 mm (0.008 in.) to 4.077 mm (0.161 in.).

Each sand was tested at three different slopes, 0.0010, 0.0015, and 0.0020, closely following the range of values used by Captain Kramer. The technique adopted for the tests was also similar to that of the paper, the general practice being to set the tilting flume at the desired slope, mould the sand to this slope, and by adjustment of the tail-gate, to maintain the slope of the water surface at this same value throughout each run, thus insuring uniform flow. Each test was started with a small flow, usually in the range of laminar flow. After an equilibrium had been reached, and a complete set of observations made, the flow was increased slightly, the tail-gate was again adjusted, if necessary, to maintain the proper slope, and another set of observations was made. This procedure was repeated, increasing the flow by small increments, until the maximum capacity of the circulating system had been reached.

The results of these studies have been assembled by the writers[54] and, therefore, it is not deemed necessary to repeat this information in the present

NOTE.—The paper by Hans Kramer, Assoc. M. Am. Soc. C. E., was published in April, 1934, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: August, 1934, by Messrs. John Leighly, Paul W. Thompson, and Gerard H. Matthes; September, 1934, by Messrs. R. H. Keays, and F. T. Mavis; and November, 1934, by Messrs. V. V. Tchikoff, Morrough P. O'Brien and Bruce D. Rindlaub, and Herbert D Vogel.

[52] Research Asst., U. S. Waterways Experiment Station, Vicksburg, Miss.

[53] Research Asst., U. S. Waterways Experiment Station, Vicksburg, Miss.

[53a] Received by the Secretary September 17, 1934.

[54] "Studies of River Bed Materials and Their Movement, with Special Reference to the Lower Mississippi River", *Paper 17*, U. S. Waterways Experiment Station. (Publication pending.)

discussion. However, the writers do wish to point out a few limitations to Captain Kramer's methods which have not before been properly emphasized.

In the first place, it has not been found practicable by the writers to utilize the visual method of determining the commencement of general movement of bed-load. This difficulty was first encountered in the early stages of the studies, when water from the artificial lake was being used in the flume. This water was often too turbid, especially after rains, to allow observations at depths greater than 2 or 3 in. After the completion of the tests on the first three sand mixtures, however, the flume was connected with a circulating clear-water system; but it still was found impossible to obtain consistent results from the visual method of determining general movement. In spite of the care with which the operators made their observations, the element of subjective judgment entered into the results, and it was frequently found that two or more skilled observers differed by as much as 100% in their individual selection of the point of general movement.

This difficulty has been overcome by the adoption of a mechanical method of spotting general movement; that is, basing its determination primarily upon analyses of the sand in motion and comparison of these analyses with that of the original mixture.

Another difficulty arose from the definition of general movement, proposed by Captain Kramer, namely (see "Definitions: Bed-Load Movement" (Definition 4)), "that condition in which sand grains up to and including the largest are in motion". Implied throughout the author's paper is the assumption that the smallest materials are placed in motion first, the medium-sized grains next, and the largest grains last. From this, it follows that the movement of the largest grains is the real criterion for the condition of general movement. Correspondence with the author has verified the fact that he used the largest grains as his criterion for movement, assuming that when they were in motion, all sizes up to the largest were also in motion. However, the writers have found that for sands of mean grain size smaller than about 0.5 mm (0.0197 in.), the large grains moved first, and that if their movement were used as a measure for general movement, an erroneous value of critical tractive force would be obtained. This curious phenomenon has been noted independently by several observers, and has been checked by sieve analyses which have been made of the sand caught in the trap at the lower end of the flume.

This condition was not observed in the two largest sands tested, the mean grain sizes of which were 4.077 mm (0.161 in.) and 0.586 mm (0.023 in.), respectively. It was observed, however, for each of the other seven mixtures, all of which were of smaller mean grain size than the finest mixture tested by the author.

Fig. 18 illustrates this condition. The mean grain size of the material in motion (as determined from sieve analyses of the trapped materials) is compared with the mean grain size of the original mixture in Fig. 18(*a*) (see the horizontal dash-dot line). The average size of the material in motion in the first five runs in this test on Sand No. 4 was much larger than that of the original sand mixture. It was only after the appearance

of riffles, at a much higher value of tractive force than the critical cited by Captain Kramer, that the materials in motion in this test became reasonably representative of the original mixture.

In Fig. 18(*c*), the curve for the rate of bed-load transportation illustrates another weakness in Captain Kramer's definition for critical tractive force.

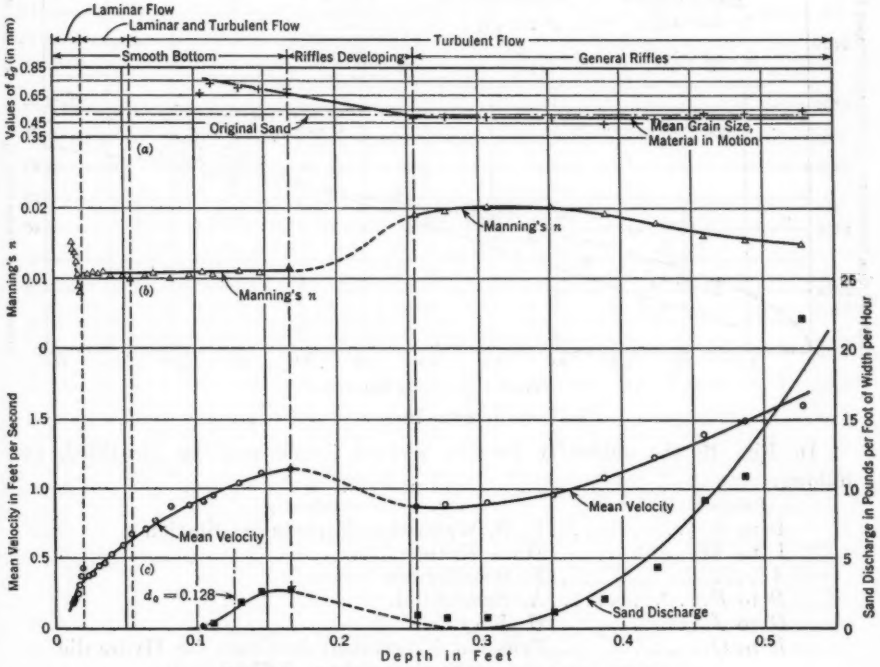

FIG. 18.—STUDY OF SAND NO. 4; SLOPE, 0.001.

With this sand, and with all the seven finest mixtures tested, there was a gradually increasing rate of movement on a smooth bottom, up to the point where riffles began to form. At this point the bed suddenly became much rougher, the velocity decreased, and the depth for a given discharge had to be increased by raising the tail-gate, in order to maintain uniform flow. The result was an appreciable slowing down of the movement of the bed-load, with a complete cessation of progressive down-stream movement in the case of the finest mixtures. The author's value for critical tractive force, however, either from Equation (10), or from his visual method, locates his "general movement" on the smooth bed; and at higher values than his "critical", there may be almost no bed-load movement at all. This fact is well illustrated in Table 9. The value, d_o, in Fig. 18(*c*), is the depth at which "general movement" obtains on the smooth bottom, according to the author's method. Fig. 19 shows the values of critical tractive force plotted against the sand characteristics. The experimental value for Sand No. 4 is seen to be 0.0084 lb per sq ft, and the value as taken from the author's curve in Fig. 19

is 0.0070 lb per sq ft. For a slope of 0.0010, the latter tractive force is obtained at a depth of 0.112 ft. Fig. 18(*c*) shows that at this depth there was practically no movement of Sand No. 4.

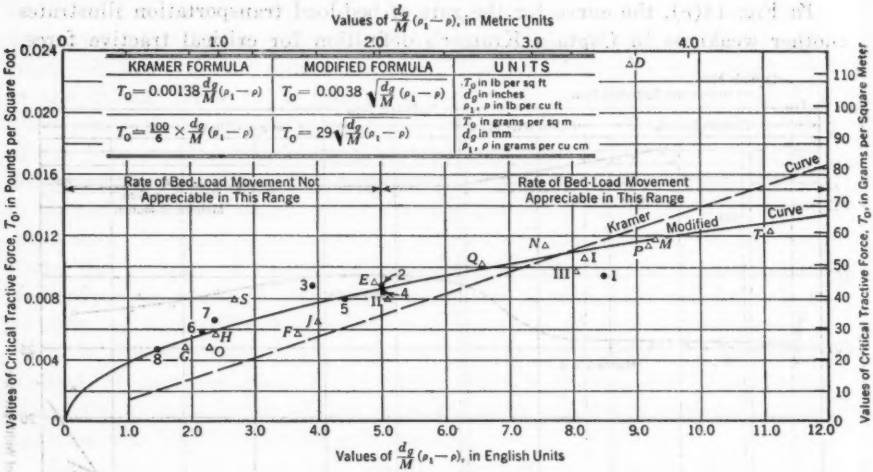
Values of $\frac{d_g}{M}(\rho_1-\rho)$, in Metric Units
KRAMER FORMULA
$T_0=0.00138\frac{d_g}{M}(\rho_1-\rho)$
$T_0=\frac{100}{6}\times\frac{d_g}{M}(\rho_1-\rho)$
MODIFIED FORMULA
$T_0=0.0038\sqrt{\frac{d_g}{M}(\rho_1-\rho)}$
$T_0=29\sqrt{\frac{d_g}{M}(\rho_1-\rho)}$
UNITS
T_0 in lb per sq ft
d_g in inches
ρ_1, ρ in lb per cu ft
T_0 in grams per sq m
d_g in mm
ρ_1, ρ in grams per cu cm
Rate of Bed-Load Movement Not Appreciable in This Range
Rate of Bed-Load Movement Appreciable in This Range
Kramer Curve
Modified Curve
Values of Critical Tractive Force, T_0, in Pounds per Square Foot
Values of Critical Tractive Force, T_0, in Grams per Square Meter
Values of $\frac{d_g}{M}(\rho_1-\rho)$, in English Units

FIG. 19.

In Fig. 19 the authority for the plotted points may be identified, as follows:

Points	Authority
1 to 8	U. S. Waterways Experiment Station
I to III	Hans Kramer
A	F. Schaffernak
B to *F*	A. Schoklitsch
G to *J*	H. Krey
K to *Q*	Prussian Experiment Institute for Hydraulic Engineering and Shipbuilding
R	H. Engels
S to *V*	G. K. Gilbert[26]

The values at the Experiment Station were obtained from an analysis of the data after the completion of the tests, and were selected to conform as closely as possible to Captain Kramer's definition. In their selection, due weight was given the visual classification of the movement noted during the test, the rate of movement, and the roughness value of the bed, and it is believed that they are consistent with the author's data. The following criticisms should be made of Fig. 19:

1.—A new curve, parabolic in shape, has been drawn to replace the straight line drawn by Captain Kramer through his points. The equation for the "modified" curve is (T_o, in pounds per square foot; D_g, in inches; and ρ_1 and ρ, in pounds per cubic foot):

$$T_o = 0.0038 \sqrt{\frac{D_g}{M}(\rho_1 - \rho)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (27)$$

[26] "The Transportation of Débris by Running Water", by G. K. Gilbert, *Professional Paper No. 86*, U. S. Geological Survey, Washington, 1914.

or (T_o, in grams per square meter; D_g, in millimeters; and ρ_1 and ρ, in grams per cubic centimeter),

$$T_o = 29 \sqrt{\frac{D_g}{M}(\rho_1 - \rho)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (28)$$

which corresponds with the author's Equation (10) for the straight line. It must be noted that in Equations (27) and (28) the coefficient before the radical no longer is dimensionless, but has the units, $\frac{M^{\frac{1}{2}}}{L^{\frac{1}{2}}\,T}$ (using the mass-length-time system).

2.—It is believed that the part of the curve to the left of the abscissa value of 5.0 (English units) has no practical value, and should not be used for any practical design work, either in hydraulic models or in regulatory works for rivers. The reason for this limitation is that sand mixtures falling within this range will riffle to abnormal heights at values of tractive force higher than those indicated by the curve, and will cause a retardation, or even a cessation, of the rate of movement at tractive force values greater than the author's "critical". Table 9 illustrates this fact.

TABLE 9.—RETARDATION OF BED-LOAD MOVEMENT AT TRACTIVE FORCES ABOVE THE AUTHOR'S "CRITICAL"

Sand No.	Mean Grain Size: Millimeters	Mean Grain Size: Inches	Uniformity modulus, M	Value of $\frac{D_g}{M}(\rho_1 - \rho)$ D_g, in inches, ρ_1, ρ, in pounds per cubic foot	Critical Tractive Force, in Pounds per Square Foot: Average experimental value	Critical Tractive Force, in Pounds per Square Foot: Value from author's curve	Critical Tractive Force, in Pounds per Square Foot: Value from modified curve	Rate of Movement, in Pounds per Foot Width per Hour (Dry Weight): At critical tractive force	Rate of Movement, in Pounds per Foot Width per Hour (Dry Weight): Least at tractive forces above critical
(*a*) United States Waterways Experiment Station									
1	0.586	0.0230	0.280	8.50	0.0094	0.0117	0.0112	2.5±	1.5
2	0.541	0.0213	0.439	5.04	0.0088	0.0069	0.0086	1.0±	1.0±
3	0.525	0.0207	0.539	3.98	0.0088	0.0055	0.0076	1.0±	1.5
4	0.506	0.0199	0.406	5.04	0.0084	0.0070	0.0086	1.0±	0.04
5	0.483	0.0190	0.438	4.49	0.0080	0.0062	0.0080	1.0±	0.10
6	0.347	0.0137	0.643	2.20	0.0060	0.0030	0.0057	0.02±	0.0
7	0.310	0.0122	0.525	2.40	0.0066	0.0034	0.0060	0.10±	0.03
8	0.205	0.0081	0.560	1.50	0.0047	0.0021	0.0047	0.04±	0.0
9	4.077	0.1605	0.566	29.20	0.0580	0.0404	0.0205	1.0	1.0
(*b*) The Author									
I	0.705	0.0278	0.358	8.27	0.0106	0.0114	0.0110	.....	.....
II	0.558	0.0220	0.461	5.06	0.0080	0.0070	0.0086	.....	.....
III	0.800	0.0315	0.414	8.06	0.0098	0.0112	0.0109	.....	.....

The experimental values of critical tractive force for Sands Nos. 1 and 2, Table 9, throw some light on the value of the uniformity modulus, M, which was proposed by Captain Kramer as a measure of voids ratio. These two sands differ by only 0.045 mm (0.0017 in.) in mean grain size, but their uniformity moduli are 0.280 and 0.439, respectively. In spite of this great difference in M-values, the critical tractive forces were nearly equal, being 0.0094 and 0.0088 lb per sq ft, respectively. Although this one fact is not sufficient evidence on which to accept or reject the use of the uniformity

modulus, M, it does indicate that it might be more reasonable to give it less weight in the equation for critical tractive force.

A preliminary investigation has been made to determine whether the value of M is a reliable measure for the voids ratio. Nine mixtures were moulded in the tilting flume, separated from each other by wooden partitions, and allowed to stand under water for some hours under like conditions. Two samples were then taken from each mixture, and the mean grain size, uniformity modulus, and voids ratio for each were determined. Table 10 contains a summary of the results of these analyses, the values for voids ratio being the average of the two values determined.

TABLE 10.—RELATION BETWEEN UNIFORMITY MODULUS (M) AND VOIDS RATIO

Mean grain size, in millimeters	Uniformity modulus, M	Voids ratio
(*a*) UNITED STATES WATERWAYS EXPERIMENT STATION		
0.566	0.30	0.61
0.668	0.33	0.63
0.557	0.42	0.68
0.505	0.42	0.71
0.349	0.43	0.73
0.547	0.44	0.65
0.216	0.53	0.85
3.761	0.54	0.65
0.240	0.58	0.86
(*b*) THE AUTHOR		
0.705	0.36	0.62
0.800	0.41	0.67
0.558	0.46	0.63

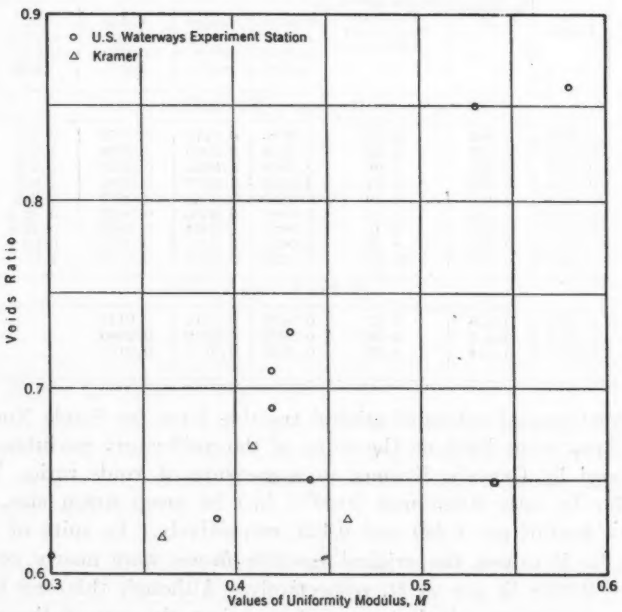

FIG. 20.

Table 10(*b*) contains values for Captain Kramer's sands, Nos. I, III, and II, respectively. These values were listed by him in his unabridged paper (record manuscript on file in Engineering Societies Library), for loose sand under water, the condition similar to that under which the nine samples were tested at the Waterways Experiment Station.

Plotting the values of M against corresponding values of voids ratio (see Fig. 20) indicates that while there may be a tendency for the one to vary with the other, the points are too scattered to show any consistent relationship.

In conclusion, it must be recognized that the author has given an impetus to the study of bed-load movement. It is gratifying to note that much thought is now being given to this subject.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

LABORATORY TESTS OF MULTIPLE-SPAN REINFORCED CONCRETE ARCH BRIDGES

Discussion

BY M. HIRSCHTHAL, M. AM. SOC. C. E.

M. HIRSCHTHAL,[12] M. AM. SOC. C. E. (by letter).[12a]—Of special interest are the effects, described in this paper, of expansion joints in the deck of the spandrel construction on the resistance of the arch ring or rib. There is no doubt that a continuous spandrel system extending from pier to pier, fixed to the arch at its various points, will increase the resistance of the arch rib or ring, to the various stresses to which it is subjected. This is natural, since the structure then becomes a braced arch or a Vierendeel truss with a curved lower chord and laterally supported top chord.

The difficulty lies in the necessity of designing each of the joints of this rigidly connected frame to resist the distortions due to the movement of the arch, particularly under the action of temperature stresses. With a drop in temperature the crown of the arch falls while the haunches rise, which action is transmitted to the deck through the spandrel columns or walls, causing distortions at their junction with the deck. These distortions increase greatly with long arch spans and high spandrel systems and require heavy sections to resist them.

The object in providing intermediate expansion joints in decks, therefore, is not simply to guard against the effect of temperature changes longitudinally, but to eliminate the stresses induced in the deck by the distortion of the arch ring or rib due to changes in temperature in that member. By allowing for free movement in the deck construction for this condition, it is not necessary to design for distortions due to this cause, but to select the proper position for the location of the joints.

There is further danger in the absence of intermediate expansion joints in long span arches, and it lies in the fact that, particularly in railroad bridges,

NOTE.—The paper by Wilbur M. Wilson, M. Am. Soc. C. E., was presented at the Joint Meeting of the Structural Division, Am. Soc. C. E., and the Applied Mechanics Division, Am. Soc. M. E., Chicago, Ill., June 29, 1933, and was published in April, 1934, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: August, 1934, by C. B. McCullough, M. Am. Soc. C. E.; and September, 1934, by Carroll L. Mann, Jr., Esq.

[12] Concrete Engr., D. L. & W. R. R., Hoboken, N. J.

[12a] Received by the Secretary October 25, 1934.

there is superimposed over the open-spandrel construction a heavy parapet as a guard against possible derailment, which then becomes a continuous girder spanning from pier to pier and subject to the foregoing distortions.

The writer wishes to cite an interesting experience in this connection: Between 1908 and 1911, the Delaware, Lackawanna, and Western Railroad Company constructed a cut-off line between Hopatcong, N. J., and Slateford, Pa., in the course of which two concrete arch viaducts were constructed—one over Paulins Kill, at Hainesburg, N, J., and the other over the Delaware River, at Portland, Pa. Both these viaducts were built without expansion joints, except at the piers, the floor systems being of arch construction so that the sections at the junction with the spandrels are quite heavy.

In 1912, preliminary to the design of the proposed structures on the second cut-off line between Clarks Summit and Hallstead, Pa., an examination was made of the parapets of the aforementioned two viaducts to determine whether intermediate expansion joints should be provided in the new structures. These older structures had been in operation a sufficient length of time to have been subjected to several cycles of temperature changes at that time. The examination disclosed vertical cracks in the parapets, particularly in those of the Delaware River Viaduct, possibly due to the fact that the arches had considerably less rise in proportion to span. Although the cracks were not serious, they indicated the effects of the omission of intermediate expansion joints, and it was decided to provide them in the then proposed structures—the Martin's Creek and Tunkhannock Viaducts. These two viaducts have been in operation for more than twenty years, and the expansion joints have functioned so as to eliminate this difficulty.

Another interesting phenomenon was observed in connection with the effect of temperature stresses on arch bridges with solid spandrel walls retaining the fill. These arches were of smaller spans, 20 to 45 ft, some of which were constructed early in the progress of the first cut-off line. Before any live load was placed on these arch bridges, the spandrel walls were found to have vertical or nearly vertical cracks at each haunch which, it was decided, were due to the distortion of the arch ring resulting from temperature changes, the high section over the haunch being subjected to greater deformation due to its height. The effect being similar to shear cracks in a beam, it was decided to place shear reinforcement in the spandrel walls in subsequent structures with the result that these cracks have not appeared. Strangely, this difficulty was not experienced with the flat over-head highway arches—probably due to the fact that the height at the haunch is much smaller and the section of the arch ring much lighter. This fact is borne out by the author's conclusion in connection with the relative effects on high and low spandrel construction.

Another point to which the writer would call attention is in connection with Professor Wilson's preliminary statement (see "Description of Specimens and Apparatus: Specimens") that "the weight of the specimen was not great enough to produce a dead-load stress commensurate with dead-load stresses encountered in the design of a full-sized bridge."

This is significant and emphasizes the care necessary in drawing conclusions from model test results and applying them to full-sized structures. The problem is intricate when the material in both model and prototype is identical; it is more so when the models are of a material having elastic properties entirely different from those of the structure; and it is particularly serious when the structure is an arch and temperature effects are to be considered. In this type of structure, the thrust and moment due to temperature change are proportional to the cube of the depth of arch section and inversely as the square of a function of the rise so that no direct relationship exists between a reduced scale structure and one of full size.

Attention is called to the summary of conclusions, showing an appreciable increase in stress in multiple-arch spans on slender piers over that in arches of single spans with fixed abutments. This should be a warning in the design of multiple-span arch structures of the so-called "rigid frame" type when computing dead load stresses in its members.

It is needless to state that the author has contributed exceedingly valuable information in a field where it is most urgently needed, and his painstaking attention to all the considerations involved cannot be praised too highly. It is to be hoped that similar tests on skew arches will be undertaken to provide the necessary information for such structures.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

THE RESERVOIR AS A FLOOD-CONTROL STRUCTURE

Discussion

BY MESSRS. F. KNAPP, AND CHARLES S. BENNETT

F. KNAPP,[9] ESQ. (by letter).[9a]—In most practical cases the requirements of design for flood-control and water-power projects, or for modern open-river navigation, are of a conflicting nature. A water power plant is required to furnish firm power and dependable output and, therefore, should have a reservoir level as high as possible at all times. The level of the flood-control reservoir, on the other hand, should be kept as low as possible to maintain sufficient storage space to accommodate great floods. In the case of river navigation a certain minimum river stage must be maintained throughout the year. Combining these requirements results in heads for the power plant that vary between wide limits, with resulting inefficiency; at times of reservoir draw-down neither firm power nor dependable output can be guaranteed.

One possibility is to combine the functions of flood control and water power by operating the flood-control reservoir independently of the power plant by means of daily pumped storage. A certain firm head is developed by means of a maximum draw-down limit, in order to reduce the variations of head for the pumps, and the turbines, as much as possible. The power for the pumps is generated at night, by steam, thus increasing the load factor and hence decreasing the cost of the generated energy from these plants. Dependable firm power and output is being supplied by the pumped storage plant during only a few daylight hours. The pumped storage plant is thus operating as a peak load and stand-by plant. The combined effect of peak-load and base-load plants, working in parallel, is to supply the energy at more favorable conditions than could be obtained by steam plants alone.

This solution, of course, is only possible when the flood-control reservoir is developed in connection, and in combination with, a large interconnected

NOTE.—The paper by George R. Clemens, Assoc. M. Am. Soc. C. E., was published in May, 1934, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: November, 1934, by C. Stanley Maxwell, Jun. Am. Soc. C. E.

[9] Asst. Hydr. Engr., The São Paulo Tramway, Light & Power Co., Ltd., São Paulo, Brazil.

[9a] Received by the Secretary September 28, 1934.

power system. A typical example of such a solution of outstanding importance is the Saale Reservoir, in Germany.[10] This development is of special interest in so far as it serves an additional purpose; namely, a provision for increasing the low-water levels of the River Elbe for modern open-river navigation. Such combined use of a single, or several, reservoirs requires a combination of the interests of the three functions involved. In the case of the Saale Reservoir the governmental authorities utilize the useful storage throughout the year for navigation purposes. At certain times (generally well known in advance) water is stored to equalize flood waves. An extensive service for reporting river levels and weather conditions serves to predict the behavior of the reservoir and to accumulate experience during the first years. The operation of the reservoir by the Power Company in combination with the demands of the governmental authorities serves to simplify and to combine the conflicting interests.

CHARLES S. BENNETT,[11] M. AM. SOC. C. E. (by letter).[11a]—No apologies should be necessary, in connection with this paper, for the elementary cataloging of the steps to be followed; this outline procedure will no doubt have wide usage because of its clear and methodical presentation. The paper constitutes the most complete and practical outline of the method of developing a study of reservoir control of floods which has come to the writer's attention.

The outline of preliminary investigations for a typical stream brings to mind the difficulties encountered in attempting to obtain definite maximum run-off data for such studies. The existing records of rainfall and run-off for most areas are usually fragmentary, even in recent years. The current interest in studies relating to the control and distribution of run-off should emphasize the value of securing continuous and related rainfall and run-off data wherever possible. It seems to the writer that it might be possible for the existing Federal agencies to co-operate to the extent of arranging for more gauging stations at points on the various streams at which the records would tie in with a group of rainfall stations on the drainage area above. It would also be useful if the records would indicate the maximum flood flows at these stations, as well as mean daily flows.

It might be implied from the author's statement under the heading, "Retarding Basin Operation", that the works of the Miami Conservancy District utilize improved channels below the retarding dams for the entire river distance below the dams. The river channels were improved only through the cities below the dams, as the flood-control plan was designed to furnish complete protection to the cities only. Through the reaches of river intervening between these cities, no improvements were made, as the cost of such work would exceed the benefits to rural property. The benefits received by the lands below the dams, but not within the cities, are only the reduction of flood peaks due to the effect of the dams.

[10] "Die Saaletalsperre," von H. Kyser, *Elektrotechnische Zeitschrift*, Nos. 28 and 29, 1933.

[11] Engr., The Miami Conservancy Dist., Dayton, Ohio.

[11a] Received by the Secretary November 12, 1934.

The retarding reservoir method of flood control is especially applicable to small streams for the local protection of lands in such areas, and such a system might have limited benefits to larger streams, such as the Ohio River or the Mississippi River, to which the controlled streams are tributary.

An advantage of the retarding-basin method of operation, from the economic standpoint, may be the fact that, in some cases, the agricultural lands within the basins need not be completely withdrawn from use, as is necessary in the case of storage basins. In the Miami Conservancy it has been possible to re-arrange the farm units in the basins and to relocate buildings on the higher sections of the new units, thus permitting continued agricultural use of the greater part of these areas. The lands originally purchased can be returned to private ownership, subject to the necessary flooding easements and restrictions. Of the 30 000 acres which were purchased by the Miami Conservancy District, approximately 20 000 acres have already been resold.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

EXPERIMENTS WITH CONCRETE IN TORSION

Discussion

By A. W. FISCHER, Esq.

A. W. FISCHER,[15] ESQ. (by letter).[15a]—The observed results obtained by the author are valuable when the member is subjected to pure torsional moment and the section is a square section. However, since practically all reinforced concrete members are not only subjected to torsion, but, at the same time, receive stresses due to bending moments, and since practically all of them are rectangular in section, the experiments in pure torsion on square sections do not help the designer for actual conditions that occur in most structural members. For that reason tests should be conducted to ascertain the combined torsional and diagonal shear on rectangular sections.

The author cites longitudinal balcony girders that support cantilever beams and exterior floor-beams. These two types certainly receive shear due to torsion and diagonal shear due to bending moments caused from vertical loads —and the depths of these members are generally from one and one-half to four times their width.

The writer does not recommend the use of inverted beams (that is, beams placed above the slab), but in some cases due to lighting conditions it is advisable to use them for the support of the outer edges of exterior slabs.

In such cases the top horizontal bars are quite essential; similar bars can be placed in balcony girders especially if the latter are designed as **T**-girders. The vertical tie stirrups must be placed as close to the outer and inner faces as permissible, and the wider the beams the more effective the stirrups will be.

If the stirrups are placed at 45° with the vertical they will be more effective, but for practical purposes it is better to place them vertically and use fairly small bars, closely spaced. Spirals will be more effective but from a practical standpoint they will not serve as well for the bottom tension steel and the top compression steel; nor can a spiral be flattened to suit a rectangular beam very easily, and, for that reason, vertical stirrups made in the

NOTE.—The paper by Paul Andersen, Assoc. M. Am. Soc. C. E., was published in May, 1934, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: August, 1934, by E. Mirabelli, Assoc. M. Am. Soc. C. E.; and October, 1934, by Messrs. Frank M. Russell, and Leslie Turner.

[15] Care, Pennsylvania Sugar Co., Philadelphia, Pa.

[15a] Received by the Secretary September 28, 1934.

form of a tie with ends having at least a hook of eight bar diameters will take care of the torsional and vertical shear in the most practical manner.

Although stirrups do not act until the beam begins to fail, if they are then brought into action so as to keep the beam from failing, they serve their purpose. A beam should be designed for the vertical and torsional shear, and the required number of vertical tie stirrups should be specified to take care of the combined shear.

It might be well for the author to demonstrate by an example, how he would design a reinforced, rectangular, concrete beam under torsion and bending moment. In this way some satisfactory method will result which some day can be verified by further experimental data.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

WAVE PRESSURES ON SEA-WALLS AND BREAKWATERS

Discussion

BY CHARLES T. LEEDS, M. AM. SOC. C. E.

CHARLES T. LEEDS,[12] M. AM. SOC. C. E. (by letter).[12a]—In bringing into small compass, and placing in usable form, a considerable part of the material on wave pressures available in many different sources, the author has performed a valuable service. As stated by Mr. Molitor, a valuable amount of observational data were collected by Lt.-Col. Gaillard who also contributed a most valuable discussion of the many phases of this problem, such as to render his book[2] one of the most important available references on this subject. Mr. Molitor has done a service in condensing and simplyfying the great mass of material there available, and he has also added valuable data from his own experience. His brief statement of certain of the formulas compiled or deduced by Colonel Gaillard, and his demonstration of their application to specific examples should be of assistance to those not thoroughly familiar with this branch of engineering.

The mathematical statements would be more satisfying if the supporting reasons were more clearly stated. For instance, the formulas for wave height in terms of fetch, as deduced by Thomas Stephenson, are:

For $D > 30$ miles:

$$h = 1.5 \sqrt{D} \qquad (20)$$

and for $D < 30$ miles:

$$h = 1.5 \sqrt{D} + 2.5 - \sqrt[4]{D} \qquad (21)$$

In the paper, these formulas are changed (see Equations (1) and (2)), and the author states that these changes were made "after introducing wind velocity

NOTE.—The paper by David A. Molitor, M. Am. Soc. C. E., was published in May, 1934, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: September, 1934, by Charles E. Fowler, Esq.

[12] Maj., Corps of Engrs., U. S. Army (*Retired*); Cons. Engr. (Quinton, Code & Hill-Leeds & Barnard), Los Angeles, Calif.

[12a] Received by the Secretary November 3, 1934.

[2] "Wave Action in Relation to Engineering Structures," by Capt. D. D. Gaillard, Corps of Engrs., U. S. A., *Professional Papers No. 31*, U. S. Corps of Engrs., 1904.

as a variable and using statute miles instead of nautical miles"; but he gives no supporting reasons or observations either for the assumed relation between wind velocity and wave height or for changing the 30-mile fetch used by Stephenson to 20 miles. It is evident that Equations (1) and (2) will give smaller theoretical wave heights than the Stephenson formulas (Equations (20) and (21)), in the case of any wind velocity less than 78 miles per hr.

The statement that "these formulas apply only to inland lakes" is not consistent with the fact that Stephenson's formulas were shown to be correct when he analyzed waves breaking on the shore of the North Sea in Scotland, of which the fetch varied from 1 mile to 165 miles.

Under the heading, "Wave Height, Wind Velocity, and Fetch," the author states: "Ocean storms are generally more or less local, and do not cover more than 50 to 100 miles." Then he states "the fetch is restricted to the storm area, which may be anything from a few miles to, say, 900 miles." These two statements appear somewhat contradictory.

When one considers the numerous physical factors of uncertain variability which may affect the dimensions of waves, it seems questionable whether any formula can be constructed by which their heights can be computed reliably. In describing certain of his observations at Port Elizabeth, South Africa, Mr. William Shield states[14] that the greatest wind velocities recorded during the three gales observed were, respectively, 60, 68, and 58 miles per hr. On the other hand, the greatest height attained by the waves on the three occasions were, respectively, 21 ft, 13 ft, and 10 ft; the fetch, depth of water, etc., were practically the same in all cases.

This apparent anomaly, he attributes to a difference in the range of the respective gales or a variation in the direction of their course (possibly far from land); or to a combination of both causes. Hence, Mr. Shield deduced that even in gales of considerable duration, a given fetch, wind velocity, and depth of water will not always produce waves of uniform height because such height depends greatly on the varying conditions of the gales producing the waves, both as regards their extent and duration in one direction.

It sometimes happens that the line of longest fetch is not the direction from which the most severe winds blow. Furthermore, the veering of the wind into a direction from which the fetch is small may sometimes produce a wheeling around of the waves, such as to create heights greatly in excess of those to be anticipated from the formula.

Indeed, waves of destructive proportions are not always due to high winds occurring in the vicinity. In several instances within recent years, heavy ground swells have occurred along the Southern California coast with a moderate amount of resultant damage. There was little or no local wind at the time, and the only explanation seems to be the occurrence of distant storms in some part of the Pacific Ocean from which no report was obtainable.

The author truly states that the only reliable data relative to the height of ocean waves must be collected by direct observations for any given locality.

[14] "Principles and Practice of Harbour Construction," by William Shield, p. 38.

This should be strongly emphasized and as there usually are few scientific observations available, the reliability of the observers must be carefully weighed.

Careful consideration must also be given to the under-water contours in the vicinity, and to the seaward or lakeward of the work proposed, as these contours may have a marked effect, both on the height and on the strength of the oncoming waves. The original design for the rubble-mound breakwater at San Luis Obispo, Calif., called for the construction of a part of its length along the crest of a submerged reef. Before its completion severe storms hammered down and drove bayward a considerable quantity of rock from the part of the breakwater that had then been constructed along the reef. When work was resumed, the following spring, the damaged part and the unfinished remainder of the breakwater were re-located a short distance back from the crest of the reef. It was found, as expected, that the wave impact on the breakwater was thereby greatly reduced.

In referring to his Equation (12) for the maximum value of the hydrodynamic pressure, Mr. Molitor states that "k is an empiric coefficient evaluated from Colonel Gaillard's observations for the Great Lakes as 1.30 to 1.71 * * *. For ocean storm waves k may be taken as 1.8." Referring to his observations at St. Augustine, Fla., Colonel Gaillard states[15] that only in one case out of 107 observations did k reach its maximum possible theoretical limit of 2.0; whereas, in three cases, it equalled or exceeded 1.8. Consequently, he declared that k *may* have a possible value as great as 2.0.

The statement in the paper that "all factors contributing to the solution of the problem herein considered [the design of sea-walls and breakwaters] are only approximately knowable," might well be further emphasized. There is too great a tendency on the part of laymen and even of inexperienced engineers to expect wave-resistant structures to be designed with the exactness and economy of land structures, such as buildings and bridges in which the forces to be resisted can be determined with a high degree of accuracy. Moreover, there are forces and contingencies which only experience can foresee and which are likely to be overlooked by even the best theorists.

An example of this fact occurred some years ago when certain plans for a concrete caisson breakwater were being reviewed by the writer. As drawn, they were pleasing in appearance and apparently showed a high degree of economy. After careful analysis, however, it was necessary to recommend the rejection of these plans, because the assumption as to the force of the waves was inadequate and because other assumptions showed a lack of practical experience on the part of the designer. Based on textbook formulas only, the design was adequate, but subsequent occurrences have demonstrated that if the breakwater had been built as designed, it would have failed.

In another instance, concrete caissons for a breakwater were actually constructed, and the first of them was floated into approximate position near its final location and then grounded temporarily on the bottom. As far as the writer knows, the caisson was well constructed, but inadequate attention had

[15] *Professional Papers No. 31*, U. S. Corps of Engrs., p. 177.

been given to the design of the breakwater foundations and to the method of placing. While the caisson was temporarily grounded on the sandy bottom, the ocean surge produced a jetting action under the caisson, washing out so much sand under one end that the caisson settled, became "hog-backed," and, under the pounding of the waves, was soon a total wreck. The same action would have taken place if the caisson had been set in its final project location. Inadequate attention had been given to the problem of foundations and the method of placing the caissons.

These instances constitute no criticism of the use of concrete caissons in breakwater construction. Too many breakwaters of this type have been successfully built and maintained to gainsay their serviceability and practicability. Their chief usefulness, like rock-filled cribs, is where rock of suitable character and size is not available or is too expensive. Where durable rock can be obtained in large pieces and at a reasonable price, a rubble-mound breakwater has the distinct advantage of simplicity of design and ease of repair.

It is simple enough to design a safe structure, if no regard is given to cost. To attain an economic engineering solution, one with an adequate but not excessive factor of safety, is not so simple. The lay mind (and many engineering minds) would dictate a structure such that no storm damage could ever occur. One may well question the economy of such procedure, however. Therein lies the advantage of a breakwater or sea-wall design such that it will resist the stress of usual storms, but which is susceptible of sustaining slight damage in an abornmal storm without complete destruction. When reduced to an annual basis the cost of such infrequent repairs is likely to be much less than the interest on the capital investment in a heavier breakwater, adequate to resist all storms.

The theoretical value for the wave pressure produced by a wave striking a breakwater obliquely is given in Equation (19). The actual pressure is likely to exceed this value, as the friction of the wave against the breakwater tends to drag back that flank of the wave front, causing a pressure more nearly normal to the breakwater face than the general line of wave front would indicate, and also producing the heaping up of the wave so familiar to engineers who have observed such occurrences.

The paper is devoted chiefly to vertical-faced breakwaters, although some consideration is given to the form of superstructure. A vertical face naturally will involve the maximum wave stress on the structure. The author points out that it is always preferable to render the wave pressure on the structure less severe by adopting a superstructure design that does not obstruct the wave in its entirety on a vertical surface. The same reasoning holds true, where feasible, of the entire breakwater cross-section, as thereby the horizontal component of the wave force will be reduced and, also, the component normal to the surface will then have less overturning or sliding effect.

This result may be accomplished, as stated, by a sloping surface, but in a concrete structure a curved profile that turns the wave back upon itself may be preferable. Still better is a step-faced type which may be said to

resist the wave shock "on the installment plan." There are numerous examples of the successful use of this principle in the construction of sea walls.

Where rock is not too expensive to prevent its use in a rubble mound, this is ordinarily the preferable type, not merely from the standpoint of economy, and because of ease of repair, but particularly from the standpoint of safety. In this case, the waves are broken and dispersed and the wave force is dissipated in a multitude of directions. If, perchance, an unusually severe storm beats down a part of such a breakwater, it makes a broader and more stable foundation, on which the rebuilt structure will be safer than before.

The author does well to call attention to the hydrostatic pressure which wave impact produces inside a breakwater structure and which is transmitted to all parts of it. This pressure, and confined air pressure, account for many of the apparently paradoxical effects produced by storms.

The three points most commonly lost sight of by the novice are: (1) Loss of weight by submergence; (2) transmission of hydraulic pressure equally in all directions; and (3) difference in pressure, dependent on whether or not the wave is breaking.

Additional emphasis and discussion might well have been devoted to the effect of the part of a wave that is not stopped by a breakwater. Not only should consideration be given to the question of whether or not ships are to be moored close to the harbor side of the breakwater, but the stability of the of the breakwater structure itself is vitally involved. It is not sufficient that "the wall structure must be capable of resisting this static equivalent [of the energy of the part of the wave stopped by the wall] without exceeding certain requirements of structural stability and safety." Many of the recorded breakwater failures have been caused primarily by the part of the wave that overtops the structure. The falling water may produce such an impact on parts of the superstructure as to shatter it, or the substructure or foundation on the lee side may be so weakened as to result in overturning.

In conclusion, the writer wishes to commend Mr. Molitor for the service he has done the profession by this contribution to the clarification and simplification of an involved subject.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

DETERMINATION OF TRAPEZOIDAL PROFILES FOR RETAINING WALLS

Discussion

BY MESSRS. E. S. LINDLEY, AND KENNETH L. DEBLOIS

E. S. LINDLEY,[3] M. AM. SOC. C. E. (by letter).[3a]—In connection with this paper it may be of interest to call attention to another nomographic solution of Rankine's formulas[4] which seems to be simpler than that presented by Professor Pippard. To allow for surcharge, the wall is calculated for an increased height. The factor by which the height is to be multiplied is a function of the angle of surcharge, and is given in plain scales.

For a wall with a vertical face and back, a single isopleth gives the solution. The fixed points for this isopleth are, on one scale, the angle of repose, and on another, the ratio of density of masonry to that of soil. The solution is to determine the ratio of thickness to height. There is no need to find cosines from tables, as in Professor Pippard's method, and the ratios to be computed are simpler.

For a wall with a vertical face and battered back the solution is in three steps; in addition to the two given factors stated in the preceding paragraph, there is the ratio of top thickness to height. The solution is to determine the ratio of bottom thickness to height.

KENNETH L. DEBLOIS,[5] ASSOC. M. AM. SOC. C. E. (by letter).[5a]—The direct solution of gravity retaining walls algebraically and by nomographic charts, as presented by the author, will help to eliminate the "cut-and-try" method. Most retaining wall stems are designed as reinforced concrete cantilever beams. Except in high walls the vertical component of the earth pressure and the weight of the stem are neglected as affecting the bending stresses but

NOTE.—The paper by A. J. Sutton Pippard, M. Am. Soc. C. E., was published in August, 1934, *Proceedings*. This discussion is printed in *Proceedings* in order that the views expressed may be brought before all members for further discussion.

[3] Wotton-under-Edge, Gloucestershire, England.

[3a] Received by the Secretary, September 8, 1934.

[4] "Panjab Irrigation Nomograms," by E. S. Lindley, M. Am. Soc. C. E., Panjab Irrig. Branch, Public Works Dept., Panjab, India.

[5] Asst. Bridge Constr. Engr., San Francisco-Oakland Bay Bridge, Oakland, Calif.

[5a] Received by the Secretary October 27, 1934.

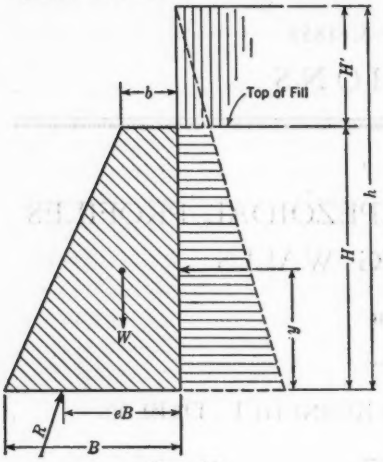

FIG. 10.

little. Many cases, however, require a gravity section for which the author's formulas and charts will apply. Railroad companies frequently use the gravity type as being more suited to supporting fills subject to heavy locomotive loads and impacts.

The equations in this paper can be extended to cover the case of a wall subject to railroad or highway live load surcharge as shown in Fig. 10. In addition to the notation of the paper, let H' = the equivalent surcharge due to live load. Then,

$$B^2 (3e - 1) + Bb (3e - 1) - b^2 = \sigma H (H + 3H') \quad \ldots\ldots(16)$$

which corresponds to Equation (10). When $e = \frac{2}{3}$:

$$2B = -b + \sqrt{5b^2 + 4\sigma H (H + 3H')} \quad \ldots\ldots\ldots(17)$$

which corresponds to Equation (11).

Formulas can be written for a similar direct solution when the back of the wall is battered. For this case, however, the writer is of the opinion that it is easier to solve each individual problem, using the numerical values.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

ANALYSIS OF CONTINUOUS STRUCTURES BY TRAVERSING THE ELASTIC CURVES

Discussion

BY MESSRS. GARRETT B. DRUMMOND, AUSTIN H. REEVES, E. G. PAULET, ADOLPHUS MITCHELL, AND DAVID M. WILSON

GARRETT B. DRUMMOND,[7] ESQ. (by letter).[7a]—The paper by Mr. Stewart is timely if for no other reason than that it emphasizes again what is the fundamental theory of indeterminate structures—the fact of continuity.

It is important to consider the limitation of the method as presented. The theory of continuity is based upon certain accepted assumptions: (1) It is assumed that the neutral axes of all members at a joint meet in a point; (2) distortions due to shear and direct thrust are negligible; (3) the relative rotation of the two ends of a member, for short lengths of the axis, is proportional to $\frac{l}{I}$ for the respective sections; and (4) the intensity and duration of the loading do not affect the elastic properties of the materials, thus permitting the assumption that for short lengths of beams the differential rotation of the two ends is directly proportional to the bending moment in the length considered.

These assumptions cause this method of analysis to be applicable in the strictest sense only to those materials which follow Hooke's law, and for steel only within the elastic limit. In structures of reinforced concrete or timber, Assumption (4) is not applicable since the ratio of stress intensity to deformation varies with both the intensity and the duration of stress. Assumption (3) does not apply to reinforced concrete.

However, it is probable that the assumption of a constant value of I in reinforced concrete beams will result in errors not exceeding 5 per cent. Such an error will not seriously affect the analysis of a reinforced concrete beam, although such limitations should be recognized.

For the determination of slopes and deflections, the writer finds it more convenient to utilize the method of stress areas. In this method, which

NOTE.—The paper by Ralph W. Stewart, M. Am. Soc. C. E., was published in October, 1934, *Proceedings*. This discussion is published in *Proceedings* in order that the views expressed may be brought before all members for further discussion.

[7] Asst. Prof. of Math., Oklahoma Agri. and Mech. Coll., Stillwater, Okla.

[7a] Received by the Secretary October 30, 1934.

obviates assumption of the beam formula, the load is the area under the curve of fiber stress in the outer fibers, divided by $E\,c$, assuming $E\,c$ to be constant. The angular change of the tangents at any two points on the elastic curve is equal to the area between the two corresponding sections on the $\frac{f}{E\,c}$ diagram. The deflection at any point is equal to moment about that point, considering the beam to be loaded with the $\frac{f}{E\,c}$ diagram.

The method of stress areas is convenient when the fiber stress is known, and the slope and deflection are desired; or when the slope or deflection are known and the fiber stress is desired.

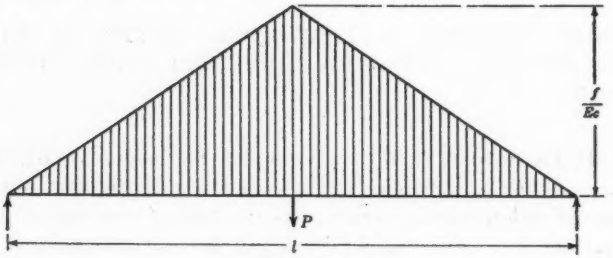

FIG. 8.

Fig. 8 represents a simply supported beam of constant cross-section, with a concentrated load at the center. The resulting curve of fiber stress will be as shown. The slope at the supports becomes $\frac{1}{4}\left(\frac{f\,l}{E\,c}\right)$, the same as the end shear with $\frac{f}{E\,c}$ as a load. The deflection at the center is equal to the moment of this load at the center, or $\left(\frac{1}{12}\right)\left(\frac{f\,l^2}{E\,c}\right)$.

AUSTIN H. REEVES,[8] ASSOC. M. AM. SOC. C. E. (by letter).[8a]—A particularly valuable feature of the method described in this paper is that it provides a complete picture of the action of any loaded structure, no matter how complicated. However, some of the problems can be solved more speedily by other methods. For example, the writer solved the problem shown in Fig. 3 within 2 min, and the one shown in Fig. 4 within 5 min, by the conjugate point method.[9]

A check by the method presented[10] by Hardy Cross, M. Am. Soc. C. E., was made on both problems shown in Fig. 6 and also on the one shown in Fig. 7, and the results given by the author were found to be exact. The writer is in agreement with the "Conclusions."

This paper is a noteworthy contribution to the knowledge of rigid frame design.

---

[8] Newark, N. J.
[8a] Received by the Secretary October 22, 1934.
[9] *Transactions*, Am. Soc. C. E., Vol. 90 (June, 1927), p. 1.
[10] *Loc. cit.*, Vol. 96 (1932), p. 1.

E. G. PAULET,[11] JUN. AM. SOC. C. E. (by letter).[11a]—A geometrical solution for the analysis of continuous frames is presented in this paper, which is termed the "traverse method." In the "Synopsis", the author states that "memorized or copied slope-deflection equations are not used, and a series of rules for the signs of moments, rotations, and deflections is unnecessary." Notwithstanding that the three basic principles stated by the author would have to be remembered by any one unfamiliar with the moment-area method, the first principle, being one of the moment-area method, is sufficient to derive the fundamental slope-deflections equations at any time. Furthermore, the moment-area method furnishes the end moments of fully restrained beams subject to any type of loading, should one not have these values immediately available for use in conjunction with the slope-deflection equations.

When the slope-deflection method first came to the attention of the writer (whose mind, to that time, had been rigidly imbued with the "work methods" expounded by Europeans), he experienced some confusion at first, due to the signs of moments, rotations, and deflections. The solution of a few simple examples of frames ended the confusion and, to this day, the writer considers the slope-deflection method as the most forceful, next to the "work methods", for the rigorous analysis of rigid frames.

The author further states, following Equation (7):

"As shown by Equation (6) the signs of the corner moments are opposite the signs of the center moment and column base moments.

"The joint rotations do not enter into the solution, but the equations are formed from the curvature units in the members."

Equation (6) furnishes no criterion for the signs of the moments, because the signs of the Δ's depend upon the relative position of the bending moment diagrams, a position which can be chosen arbitrarily and erroneously. To visualize the elastic line of the deflected simple frame and write the Δ's with their proper signs in the equation-types, Equations (6) and (8), may be an easy task for the experienced engineer; however, a complex frame may present an untrue picture to him and leave him with a feeling of false assurance as to the correctness of the results.

In this respect, the traverse method has not as direct an approach to a problem as the slope-deflection method, the equations of which will furnish the correct signs and values of moments, rotations, and deflections, the signs being in accord with those adopted for deriving the equations.

The first and second principles of the traverse method will be helpful in finding the deflection of one end of a member, or series of members, relative to the other end, after the frame has been solved by the moment-distribution method.

Fig. 5 shows the application of the author's method to a beam of variable moment of inertia, and with a settled support. In the writer's opinion, problems of this kind are more directly solved by the moment-area method.

[11] Bridge Designing Engr., State Highway Dept., Montgomery, Ala.

[11a] Received by the Secretary November 10, 1934.

Assuming that the correct arrangement of bending moment diagrams for a loaded continuous beam of uniform moment of inertia is established, the traverse method lends itself to a simple and rapid solution of such a problem.

In Fig. 6(*c*), the closing line of the "pennant" diagrams (Lines 2-1-4-8, 8-4-15-30, etc.) of the unloaded spans intersects the base line at a point of zero moment, through which (after the moments over the supports of the loaded span have been computed by the traverse method and platted to any convenient scale), the closing line of the bending moment diagrams for the unloaded spans may be drawn directly, and the moment at any point read to that scale.

From Fig. 6(*c*), it is noted with interest that the series for the relative moments and end slopes may also be found, beginning from the left end of the continuous beam, with the relations:

$$M_n = 2\ M_{n-1} + S_{n-1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (15)$$

and,

$$S_n = 3\ M_{n-1} + 2\ S_{n-1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (16)$$

in which, M_n = relative moments at the support under consideration, S_n = relative end slopes at that support; M_{n-1} and S_{n-1} = relative moments and end slopes, respectively, at the support preceding the one under consideration, and, at the first support, $M_1 = 0$, and $S_1 = 1$. For the case shown in Fig. 6(*d*), M reads S, and S reads M, in Equations (15) and (16).

ADOLPHUS MITCHELL,[12] JUN. AM. SOC. C. E. (by letter).[12a]—The method of analyzing continuous structures described in this paper, appears to hold no advantage over the slope-deflection method. On applying the two methods to multiple-story frames, one finds that whereas the slope-deflection method yields a simultaneous equation for each joint, the traverse method yields a simultaneous equation for each joint and member. The result is that an already large number of equations is doubled. Even in the simple problems solved by the author, the writer finds the slope-deflection method the easier.

Most designers are in the habit of assuming that the available foundation either offers no restraint or that it offers full restraint (fixed). As it is usually known that neither of these extremes is the case, the designer might wish to make some intermediate assumption. This can be done by comparing the geometry of the traverse for hinged and fixed conditions at End A of any member, AB. If f_{AB} is the percentage of fixation at End A:

$$\Delta_{BA} = \frac{2}{f}\ \Delta_{AB} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (17)$$

$$\theta_{BA} = \frac{1}{6}\ (4 - f)\ \Delta_{BA} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (18)$$

[12] Senior Draftsman, Bridge Dept., State Highway and Public Works Comm., Raleigh, N. C.

[12a] Received by the Secretary November 14, 1934.

and,

$$\theta_{AB} = \frac{1}{3}\ (1 - f)\ \Delta_{BA} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (19)$$

For a hinged terminal at A, $f = 0$, and for a fixed terminal at A, $f = 1$.

The author has solved the problem illustrated by Fig. 3, assuming the column terminals to be fixed. Suppose the column terminals are 50% fixed and that it is desired to determine the corner moment. Applying Equations (17), (18), and (19), $\Delta_2 = 4\ \Delta_1$ and $\theta_{21} = \frac{7}{12}\ \Delta_2 = \theta_{34}$. Since M_2 must be equal to M_3, $\Delta_2 = \Delta_3$. Equating ordinates at the right end of Beam 34:

$$\frac{7}{12}\ \Delta_2\ l + \Delta_2 \left(\frac{2}{3}\ l\right) - \frac{P l^2}{8\ EI} \left(\frac{l}{2}\right) + \Delta_2 \left(\frac{l}{3}\right) = 0$$

Solving for Δ_2,

$$\Delta_2 = \frac{3\ P\ l^2}{76\ EI} = \frac{M_2}{EI} \left(\frac{l}{2}\right)$$

and, hence,

$$M_2 = \frac{3}{38}\ Pl$$

This idea of fixation or restraint is the basis of the method introduced by T. F. Hickerson,[13] M. Am. Soc. C. E., which the writer prefers to that proposed in this paper. In most of the papers on this subject complete tables are given permitting application to members of variable moment of inertia with relative ease. The author presents no such tables and, thereby, his method suffers another handicap.

DAVID M. WILSON,[14] ASSOC. M. AM. SOC. C. E. (by letter).[14a]—Methods of analyzing continuous structures which are fundamental to an understanding of this important subject may be divided into two classes: (1) The Maxwell-Mohr Method of Work; and, (2) Special Methods, such as (*a*) Moment-Area Method; (*b*) Slope-Deflection Method; and (*c*) Moment-Distribution Method (Cross Method).

The Maxwell-Mohr method of work is based upon the principle of the conservation of energy. It is the most general method available and may be used in all cases.

The special methods in the foregoing classification apply only in the analysis of straight structural members in which bending is the cause of the primary deformations, those due to shear and direct stress being neglected. They may be developed from the method of work, or they may be derived independently. The special methods have extensive application in practice because they are convenient to use. However, one should remember that, in many cases, deformations due to shear and direct stress cannot be neglected safely and, therefore, the results obtained by special methods will not be significant.

[13] "Structural Frameworks", by T. F. Hickerson, Univ. of North Carolina Press, 1934.
[14] Associate Prof. of Civ. Eng., Univ. of Southern California, Los Angeles, Calif.
[14a] Received by the Secretary November 17, 1934.

The author has presented a method of analyzing the moments in the members of continuous frames which is based directly upon the principles of the moment-area method. It is of special interest because it pictures the approximate deformations of the structure being studied.

In general, every structural engineer has his own favorite schemes of analysis. The method proposed will undoubtedly be of value to the designer who uses the moment-area method in preference to all other methods, wherever it is applicable.

The moment-distribution method is the most workable of the special methods because the solution of simultaneous equations is not required. Furthermore, the unknown moments are determined directly without first solving for rotation angles as required by the slope-deflection method, or for delta angles as required by the author's method.

In many cases, a combination of the special methods is desirable. For example, the fixed-end moments used in moment distribution may be determined conveniently by the application of the principles of area moments. At the same time, a thorough knowledge of slope deflection is of great value in understanding the steps in moment distribution.

In order to compare the proposed method with the moment-distribution method, the writer solved the problem shown in Fig. 7 by each method. Referring to Fig. 7(*c*), the following equations from the author's method may be written since the vertical deflection of the beam at any support with reference to an adjacent support is zero:

$$24\ (A_1 - \tfrac{2}{3}\ \Delta_1) - 16\ \Delta_2 + 12\ A_2 - 8\ \Delta_3 = 0 \ldots\ldots\ldots (20)$$

and,

$$24\ (A_2 - \tfrac{2}{3}\ \Delta_4) - 16\ \Delta_3 + 12\ A_2 - 8\ \Delta_2 = 0 \ldots\ldots\ldots (21)$$

Furthermore, from Fig. 7(*b*), $\Delta_1 = 7\ M_1$; $\Delta_2 = 12\ M_1$; $\Delta_3 = 12\ M_2$; and, $\Delta_4 = 9\ M_2$. Therefore, $\frac{\Delta_1}{\Delta_2} = \frac{7}{12}$, and $\frac{\Delta_4}{\Delta_3} = \frac{3}{4}$.

Eliminating Δ_1 from Equation (20) and Δ_4 from Equation (21); substituting for A_1, A_2, and A_3 their respective values; and solving the resulting equations: $\Delta_2 = 1\,164\,000$, and $\Delta_3 = 1\,112\,400$. Therefore, $M_1 = \frac{\Delta_2}{12} = 97\,000$ ft-lb, and $M_2 = \frac{\Delta_3}{12} = 92\,700$ ft-lb.

It is to be noted that the unknowns determined by solution of the simultaneous equations are not the desired moments. In a problem involving a large number of unknowns, the work necessary to solve the simultaneous equations would be prohibitive. On the other hand, the complete solution of an identical problem by the moment-distribution method is a relatively simple process,[15] in which the necessary work is approximately proportional to the number of unknowns.

[15] *Transactions*, Am. Soc. C. E., Vol. 96 (1932), p. 1.

There is no one simple method that can be applied in analyzing all continuous structures. Furthermore, no analysis is ever absolutely exact. Certain assumptions, based upon the elastic theory of structures, must be made before a problem can be solved by any of the available methods. The validity of these assumptions must be considered carefully in interpreting the results of the analysis. This requires sound engineering judgment at all times. The problem, therefore, is definitely one for the trained engineer with a comprehensive understanding of the behavior of structures under stress.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

ECCENTRIC RIVETED CONNECTIONS

Discussion

By MESSRS. CARLTON T. BISHOP, JAMES R. BOLE, ALBERT WERTHEIMER, KENNETH L. DEBLOIS, JONATHAN JONES, ARMIN ROZMAN, AND A. E. R. DE JONGE

CARLTON T. BISHOP,[2] M. AM. SOC. C. E. (by letter)[2a].—The alignment charts proposed by Mr. Dubin satisfy a long-felt need, because they provide a simple and direct method of determining the required number of rivets in an eccentric connection without assumption. His formulas are based on the usual method of calculation, and his derivations are sound.

The writer has checked the problems in his files by means of the charts with complete satisfaction. Problems with one or two rows of rivets seemed in perfect agreement, while the few tests made for three and four rows were well within the 5% which the author mentions. In all cases, the results were sufficiently accurate for practical use. Should greater accuracy be desired, larger charts may be constructed. The results obtained from the given charts are as accurate as is consistent with the assumption that the vertical components due to shear are distributed equally among the rivets, whereas the vertical components due to moment are not. This assumption is consistent with the usual theory that a concentric load is distributed equally among a group of rivets, notwithstanding the fact that absurd results may be obtained when it is applied to extreme and impractical problems.

The use of these charts will create a demand for additional similar charts, and individuals will doubtless make them to meet their own requirements. For example, a horizontal distance, $w = 5\frac{1}{2}$, for two rows of rivets, and $w_1 = 5\frac{1}{2}$ for the inner distance for four rows, would conform to the usual rivet spacing in columns and probably would meet with greater favor than the 3-in. distance. Similarly, it may be desirable to use a vertical distance other than 3 in.; it is often better to spread the rivets to $3\frac{1}{2}$ or 4 in. than to add more rivets.

NOTE.—The paper by Eugene A. Dubin, Esq., was published in August, 1934, *Proceedings*. This discussion is printed in *Proceedings* in order that the views expressed may be brought before all members for further discussion.

[2] Associate Prof., Structural Eng., Yale Univ., New Haven, Conn.

[2a] Received by the Secretary August 28, 1934.

In this connection attention should be called to similar charts by Odd Albert, Assoc. M. Am. Soc. C. E.,[3] which may be used for different spacing, both horizontal and vertical. For rivets in a single row the latter chart may be used to determine the number of rivets directly, but the chart for two or more rows can be used only for an assumed number of rivets. Even so, the correct number can be found more readily than by calculation.

JAMES R. BOLE,[4] JUN. AM. SOC. C. E. (by letter)[4a].—An interesting and useful solution of the problem of determining stresses in riveted connections, subjected to eccentric loads, is presented in this paper. The writer does not believe, however, that alignment charts are best suited for a graphical solution to the cases of three and four rows of rivets. Fig. 5, for example,

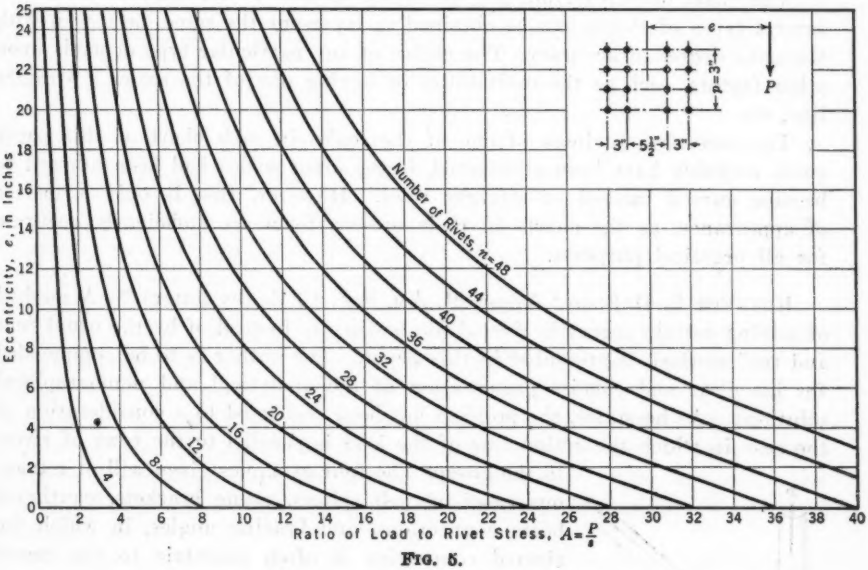

FIG. 5.

gives a solution, by means of a family of curves, for a case of four rows of rivets, in which $w_1 = 5\frac{1}{2}$ and $w_2 = 3$. This would be the standard spacing of rivet lines in the flange of a heavy column.

By substituting Equations (18) and (19) in Equation (17), the following result is obtained,

$$A = \frac{n\,M}{\sqrt{[e\,(n-4)\,p]^2 + [4\,e\,(w_1 + 2\,w_2) + M]^2}} \quad \ldots\ldots\ldots (23)$$

in which, $M = (w_1)^2 + (w_1 + 2w_2)^2 + \frac{1}{24}\,(n^2 - 16)^2\,p$.

[3] *Civil Engineering*, February, 1931, p. 413.

[4] Structural Designer, Davies & Walles, Long Beach, Calif.

[4a] Received by the Secretary September 24, 1934.

Fig. 5 was constructed by solving Equation (23) for various values of e and n. Charts of this type have several advantages over those presented by Mr. Dubin. Their construction does not require knowledge of alignment charts, or of the special method mentioned in the paper; they are more accurate and more compact.

ALBERT WERTHEIMER,[5] ESQ. (by letter).[5a]—The manner in which the complicated equations of this paper are finally presented in the form of alignment charts is of special interest. The method is quite general, in that it can be applied to any set of data in three variables, given either in tabular form, or as a family of curves. The general theory and application of this method have been discussed by the writer elsewhere.[6]

The chart obtained from any given set of data is not unique; as a rule, several types of charts can be obtained to represent the same data even with the same degree of accuracy. The choice of any particular type depends upon other factors, such as the desirability of having one of the scales a straight line, etc.

The excessive waviness of one of the scales in each chart of this paper could probably have been eliminated, if the other scales had been allowed to become curved instead of straight lines. However, this is only a matter of appearance, as the charts in their present form are sufficiently accurate for all practical purposes.

KENNETH L. DEBLOIS,[7] ASSOC. M. AM. SOC. C. E. (by letter).[7a]—A method of solving certain eccentric riveted joints directly instead of by the usual "cut and try" method, is presented in this paper. The author is to be commended for his clear and concise presentation of the analytical and nomographical solutions. As he states, the problem has been restricted to a consideration of the case in which the action line of the load is parallel to the rows of rivets in the group. The four examples given will fit numerous cases of web splices, crane brackets, cantilever beam connections, and bracing angles, in which the riveted connection is often eccentric to the center of gravity of the member.

FIG. 6.

In bridge truss design the author's examples might be of use in the top chord joints where the line of action of the load is usually eccentric to the rivet group. Practical considerations make the direct solution of heavy members more difficult, due to the different spacing of rivet lines and the number of rivets per row, to staggering, and to the unsymmetrical arrangement of rivets. The case may occur in which

[5] Care, Bureau of Ordnance, U. S. Navy Dept., Washington, D. C.

[5a] Received by the Secretary October 9, 1934.

[6] "The Graphical Transformation of a Family of Curves into Straight Lines and the Construction of Alignment Charts", by Albert Wertheimer, *Journal*, Franklin Inst. (Publication pending).

[7] Asst. Bridge Constr. Engr., San Francisco-Oakland Bay Bridge, Oakland, Calif.

[7a] Received by the Secretary October 11, 1934

some rivets are in single shear, and the others in bearing, in the same joint. Each joint of a heavy truss is a special problem and the "cut and try" method is probably the most convenient. When the author's charts do not apply and when it is desired to find the polar moment of inertia, Σr^2, of a symmetrical joint, Equations (3), (10), (16), and (22) afford a quick method

Example 1 can be extended to cover the case as shown in Fig. 6. The author's nomenclature is used with the addition of the force, F, normal to the line of action of the rivet row. In Equation (24) F and P are components of the load, R:

$$s^2 = \left(\frac{P}{n}\right)^2 + \left(\frac{F e r_0}{\Sigma r^2} + \frac{F}{n}\right)^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (24)$$

Substituting values of r_0, Σr^2, and S, as given in Equations (2), (3), and (6), respectively, Equation (24) is convertible to the form,

$$\frac{1}{A^2} = \frac{1}{n^2} + \frac{F^2}{P^2}\left(\frac{e}{S} + \frac{1}{n}\right)^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (25)$$

corresponding to Equation (7). In a similar manner, Example 1 can be extended to two, three, and four rows of rivets.

JONATHAN JONES,[8] M. AM. SOC. C. E. (by letter).[8a]—The topic of this paper has been treated in the Handbook[9] of the American Institute of Steel Construction, the solutions in both sources being restricted to certain frequently occurring special cases, for each of which a general formula is given.

The principal difference is that the author presents alignment charts for ready solution of the several formulas, whereas, the Handbook presents tables, which, for values of eccentricity other than the exact values tabulated, require interpolation. It must be granted that interpolation is sufficiently accurate for the purpose; the choice becomes, then, one of individual preference, and the writer feels that most draftsmen will prefer the tables, which are an extension of similar tables long since published in handbooks on steel construction.

There is one distinct advantage, however, in the Handbook table over the author's chart for the case of four rows of rivets, when the most usual application is considered; that is, in eccentric connections to building columns. The author uses 3 in. for the distance between gauge lines, which distance is not practical for column flange detailing. In the Handbook tables, the distances between gauge lines are those which have been standardized by the fabricating industry for column detailing. It is believed, therefore, that the presentation in the Handbook, is a superior one for the usual structural drafting-room.

ARMIN ROZMAN,[10] ESQ. (by letter).[10a]—The method of developing alignment charts and formulas for eccentric riveted connections presented by the author is a valuable contribution to the field of designing details. The con-

[8] Chf. Engr., McClintic-Marshall Corporation, Bethlehem, Pa.

[8a] Received by the Secretary October 16, 1934.

[9] "Steel Construction," 1934 Edition, pp. 222-223.

[10] Asst. Structural Engr., U. S. Treasury Dept., Washington, D. C.

[10a] Received by the Secretary October 16, 1934.

struction of this type of chart becomes comparatively simple and rapid. In his paper, Mr. Dubin presents the most commonly used charts which only need to be copied for application. Other eccentric problems can be easily solved by his method.

For example, consider the detail of bracket connections, such as those shown in Fig. 7. The stress in the extreme rivet is the resultant of the

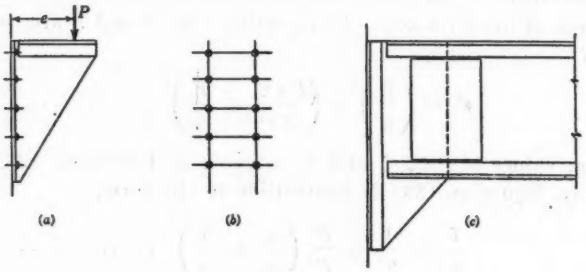

FIG. 7.

vertical shearing stress and the moment-produced horizontal stress, or tensional stress. Theoretically, the computation of this horizontal stress is quite complicated. The moment is resisted by couples composed of concentrated tensional stresses on the rivets and uniform compression stresses, depending on the strength of the bracket itself. The problem is somewhat similar to that of reinforced concrete beams. It involves too many unknowns to be solved by a method practical for ordinary usage and for designing charts. In practice, however, it is assumed that the moment is resisted by couples acting around the gravity center line of the rivets, the couples being composed of compressional as well as tensional concentrated stresses on the rivets. As a result, the connection is stronger than necessary, and the "short-cut" is on the safe side.

The formulas will be similar to Equations (1) to (4) in Example 1: Equation (1) remains unchanged; Equation (2) becomes:

$$r_o = p\,\frac{\left(\frac{n}{2} - 1\right)}{2} = \frac{p\,(n-2)}{4} \qquad \ldots\ldots\ldots\ldots\ldots (26)$$

Equation (3) becomes:

$$\Sigma r^2 = \frac{p^2 n}{48}\,(n-2)\,(n+2) \qquad \ldots\ldots\ldots\ldots\ldots (27)$$

and Equation (4) becomes:

$$\left(\frac{1}{A}\right)^2 - e^2\left[\frac{12}{pn\,(n+2)}\right]^2 = \frac{1}{n^2} \qquad \ldots\ldots\ldots\ldots\ldots (28)$$

With a standard value of p, Equation (27) can be solved by an alignment chart.

In the design of plate-girder web splices, Fig. 1 cannot have as general use as would appear on first thought. For illustration examine first a simple girder with one concentrated load (see Fig. 8(*a*)). At any section the moment is equal to the shear (in this case, R_1) times the arm, x, which

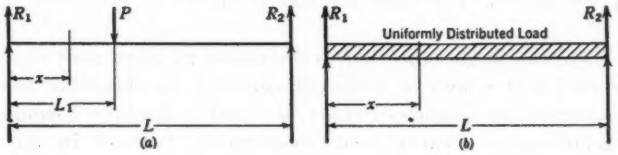

FIG. 8.

varies from 0 to L_1. Next, examine a simple girder with uniform load (see Fig. 8(*b*)). At any section the moment is equal to the shear times the arm. The shear is equal to the resultant of all the forces to the left of the section, or $R_1 - ux$, and the point of action of the resultant force is outside the span. For a section taken at the center of the span the point of action of the resultant force is at infinity and the force is equal to 0.

Similar reasoning can be offered for other types of loading, and, therefore, the arm can never be less than the distance between the splice and the support (as in the case of one concentrated load), and its maximum is infinity (as in the case of uniform loading).

The moment is maximum where the shear is zero, and it follows that at any point of maximum moment the arm is infinitely long; that is,

$$\frac{\text{Moment}}{\text{Shear}} = \text{arm}; \quad \frac{\text{Maximum moment}}{0} = \infty$$

A plate girder web is usually spliced if the girder is too long to be fabricated or to be shipped in one piece, and then it is spliced somewhere between the third point and the center. A girder less than 40 to 50 ft long is seldom spliced, and, therefore, a chart to meet the foregoing requirements must have as its limits of eccentricity about 15 ft to infinity, which does not seem practical. A chart for web splices, however, can be utilized profitably in designing wind-bracing brackets of plate girders, in which case it is commonly assumed that the moment is equal to a vertical concentrated load times one-half the span. The eccentricity is always one-half the span. Moreover, since generally the vertical shear in the girder due to the wind forces is neglected, Equation (4) and, consequently, Fig. 1 can be simplified. Naturally, the splice must be investigated independently for the shear due to the gravity load.

Care should be exercised, however, in computing the area, A, when using the chart, because s is not the allowable rivet value as in the case of simple eccentric connections, but depends on the location of the extreme rivet as compared with that of the extreme fiber.

A. E. R. de JONGE,[11] M. AM. SOC. C. E. (by letter).[11a]—The subject of eccentric riveted connections has already been dealt with many times by numerous investigators.[12] Thus, the subject is not a novel one.

As the author states at the beginning, he has set himself the task of eliminating the "cut-and-try" method necessitated by the "families of curves" used by his predecessors. For these families of curves he substitutes an alignment chart.

In itself, there can be no objection to the use of an alignment chart if it is easily constructed and allows of sufficient accuracy in obtaining the results. In order to construct an alignment chart, the author develops formulas which express the relationship between load, eccentricity, "stress" in the extreme rivet, spacing of rivets, and number of rivets in the group. At this place the critical reader misses the diameter of the rivet which, of course, is another important variable in this problem.

Furthermore, the author restricts himself to the case of a force, the line of action of which is parallel to the rows of the rivet groups, and to a rivet spacing (rivet pitch) of 3 in. In the case of two rivet rows, he considers a spacing of the rows from 3 to 10 in., while in the case of three and four rows he deals with a 3-in. spacing between rows only. Thus, it follows from the beginning that the author has covered only a few of the possible cases, so that his alignment charts require considerable extension before they can cover all the cases that may be solved by the families of curves used by his predecessors. In fact, he too would require families of curves to cover all cases possible, and, in the alignment charts, these families of curves would be far more complicated (note the wavy curves and the overlapping w-curves for various values of n in the examples treated) than those of his predecessors. Besides they would make the reading off of the values much more troublesome, quite apart from the far greater amount of labor required in constructing these charts.

Moreover, alignment charts are simple and accurate only when they consist of straight lines. The mere fact that the curves must actually be drawn involves errors, which it is impossible to predict and to guard against, and considerably complicates the task of laying out these charts. Undoubtedly, the severity of this task would have become more evident had the author described how he actually arrived at the curves and constructed the charts, and since he has covered only a small fraction of the possible cases, this description of the process of construction is the very subject that is of a far greater interest to the average engineer than the few ready charts included in the paper.

Another fact that should not be overlooked is the accuracy of the charts for certain conditions. For example, in Fig. 1, let the ratio of the load to the rivet stress be 4.5, while the eccentricity be 2 in.; then, the intersection between the n-curve and the straight line joining the two points on the

[11] With Babcock & Wilcox Co., New York, N. Y.

[11a] Received by the Secretary November 13, 1934.

[12] See "Rivet Joints; An Historical Survey of Their Development, with a Bibliography and Abstracts from the Literature", by A. E. R. de Jonge, Research Publications, A. S. M. E. (1934).

respective guide scales is so flat that the slightest deviation from the true location, due to errors committed in laying out or drawing the curves, changes the result considerably. In this case, the uncertainty of the result lies between five and seven rivets. The same holds for the other charts in similar cases, as, for example, in Fig. 2 for an eccentricity of 2 in. and a ratio of load to rivet stress of 3, in which case the uncertainty of the spacing between rivet rows is between $w = 6$ and $w = 9$ in. for $n = 4$ rivets. Similar conditions prevail in the remaining charts of Figs. 3 and 4. Thus, it cannot be said that the accuracy in the cases cited is great, an obvious drawback to the use of the alignment charts. Further disadvantages are the uneven increments of the various scales (since they likewise affect the accuracy of the alignment charts) and the necessity of keeping a ruler handy when using the charts.

Furthermore, the author's use of the term, "total stress", in the extreme rivet is unfortunate because stress is now understood to denote what was formerly called "unit stress". In fact, the author obviously means "load on the extreme rivet."[12b] This ambiguity could have been avoided easily if, in the list of notations, the dimensions of the various quantities had been stated definitely. On the other hand, the introduction of the "load" on the extreme rivet is the reason why the rivet diameter, this rather important variable, does not appear in the author's formulas and charts. The very objection to the older methods which the author intends to eliminate, namely the "cut and try" method, is thus re-introduced by him in that an assumption must be made with regard to the diameter which, if it should prove unsuitable, would have to be amended by a different choice. This, however, is nothing but a "cut and try" process.

There remains then the setting up of the alignment charts which, for Example 1, can be accomplished readily by the perusal of any treatise on graphic representations and alignment charts, although it would be necessary for the reader to obtain such a treatise before he was able to construct the chart, unless he is already conversant with the methods used.

In the case of Examples 2 to 4, the author admits that even such treatises would not help the reader, as there does not as yet exist a standard method that would readily allow of the construction of the alignment charts which represent the respective formulas. In fact, the author refers to a method by A. Wertheimer, of the Bureau of Ordnance, United States Navy Department, which, by a process of successive approximations, allows of setting up such an alignment chart. Unfortunately, the author has given no reference as to where the description of this method may be found; nor has he stated whether it has been published at all. Under these circumstances it is neither possible to check the charts for Examples 2 to 4, nor to make use of alignment charts for cases other than those covered by the charts given in

[12b] A serious difficulty arises from the author's use of the term, "total stress", in connection with the denomination, "sq in.", as given in Figs. 1, 2, 3, and 4 in the sub-caption. This combination makes it appear that the author has used the term, stress, as "unit stress" and has set up non-homogeneous equations. The writer notes that the denomination, "sq in.", given in Figs. 1, 2, 3, and 4, should be deleted; for example, in Fig. 1, the sub-caption should read "(*b*) for $A < 5$."

the paper. Thus, it would be of value to the profession if these omissions were rectified by the author.

After having discussed the shortcomings of the paper with regard to detail it seems appropriate to review it also from a more general point of view.

What has the author really accomplished? What new and permanent good has his paper brought the profession?

The answer to these questions can only be that he has introduced a new method of plotting results that are already known. As far as advancing the theory of structures is concerned, the author does not bring forward any new points since his assumptions are the same as those used by all his predecessors, namely: (1) A perfectly rigid plate; (2) an even distribution of the direct load over all the rivets of the joints; (3) direct contact between rivet shank and plate; (4) all the rivets act in shear; and, (5) deformation of the rivet shanks in proportion to their distance from the center of gravity of the rivet group.

It is well known that none of these assumptions holds true in practice. The plate is not perfectly rigid; and the rivets of the joint do not carry an equal share of the direct load, but the outermost rivets of the group (in the direction of the line of action of the force) carry by far the greater proportion of the load as was clearly shown by A. Hrennikoff in his paper[12] and in the discussion which ensued, particularly by the literature cited at that time by the present writer. Furthermore, the rivet shanks do not bear against the plate at all, at least not if the connection has not been over-strained previously so that permanent slip and deformation have occurred. This is due to the fact that, at working loads, the riveted joint carries the load by the frictional resistance developed between the plates. Finally, the bending moment distorts the outer parts of the bracket or connection plate more than the inner parts, due to the greater stresses which act in the outer parts. Thus, the assumption that the rivets, in supporting the bending moment, carry a load in proportion to their distance from the center of gravity of the group is likewise incorrect, the outer rivets supporting, in this case, too, the greater part of the moment.

It is perfectly clear, therefore, that the outer rivets carry by far the greater part of the eccentric load, contrary to the results shown by the older methods of calculation, including that by the author's alignment charts. Thus, these charts do not give the correct loading that comes on the rivets, and the paper can only be of academic interest as establishing a new method of presenting, in a new mathematical form, data already known to be incorrect.

In this connection, a few words on the excessive use of mathematics in cases where it is not essential, are in order. It is a great pity that, at present, many investigators take up only the mathematical side of engineering problems without duly considering the physical (and practical) foundations underlying them. Many highly complicated treatises are written nowadays based on greatly simplified assumptions, thus being of doubtful

[12] "Work of Rivets in Riveted Joints", by A. Hrennikoff, *Transactions*, Am. Soc. C. E., Vol. 99 (1934), pp. 437–489.

value in practice. This is particularly the case in the theory of structures, and a simple reflection often shows that the results arrived at must be fallacious on that very account, as they do not take into consideration the underlying physical conditions which, generally, are of a very much more complex nature than the simplified assumptions made.

To give an example: What could be simpler than an ordinary pin such as is often used in bridge or truss construction. It is generally assumed that it is subjected to a bending moment and shear, the former giving a stress generally far in excess of the shear stress. If, however, instead of the Navier theory, which makes simplifying assumptions, the more correct mathematical theory of stress is used, as was done by Friedrick Bleich,[14] it is found that the stresses near the ends of the pin are far in excess of both those due to bending and those due to shear. Another example is the distribution of stress at points of fixation, a problem which only in quite recent times is beginning to be studied.[15] Furthermore, in all problems it is assumed that the material is perfectly homogeneous and isotropic, a requirement which no engineering material fulfills. A little blow-hole or inclusion may alter the calculated stresses far beyond the yield point. These are all facts that are often completely overlooked; and yet, it is the correct balance between the physical foundations of a problem, the simplifying assumptions, the mathematical treatment (graphical or analytical treatment), and the accuracy in calculation (which often is driven unnecessarily far beyond requirements), that constitute a practically useful solution of a problem.

A further example of the trend of the times is the following: When the writer recently was asked for advice by a University Research Fellow regarding a certain stress problem, and when he suggested that a graphical solution might yield results of sufficient accuracy for practical purposes where the analytic methods would give either no results, due to the complexity of the problem, or results which (due to the simplifying assumptions necessary to solve the differential equations) would not be in conformity with the actual stresses obtaining in the problem, the gentleman in question concurred readily that a graphic treatment might yield a result, but stated that such a solution would not be as highly scientific (meaning mathematical). This answer is typical of the attitude often found.

Lord Kelvin is credited with having stated that "a problem is never solved until it has been reduced to its simplest terms". To this must be added that its foundation must be sound and should be investigated first.

It is often more necessary for engineers to display sound common sense (as witness the early pioneers who, with their very incomplete knowledge of the theory of stresses, erected buildings that have stood the test of time), than to rely on calculations of doubtful value. In this respect, the writer

[14] "Der gerade Stab mit Rechteckquerschnitt als ebenes Problem", von Friedrich Bleich, *Der Bauingenieur*, Vol. 4, pp. 255–259, May 15; pp. 304–307, May 31; pp. 327–331, June, 1923. See, also, his "Theorie und Berechnung eiserner Brücken", Chapter 4, Paragraph 15, Julius Springer, Berlin, 1924.

[15] For example, "Der Einfluss von Einspann-und Kraftangriffsstellen auf die Dauerhaltbarkeit der Konstrukionen", von A. Thum and F. Wunderlich, *Zeitschrift* für Verein deutscher Ingenieure, Vol. 77, No. 31, pp. 851–853, August 5, 1933.

recalls Sir Frederic Bramwell's definition of "engineering" as: "The art of drawing sufficient conclusions from insufficient premises". In fact it is often better to rely on such conclusion drawn by common sense than on mathematical derivations which are based on insufficient premises. If this would be constantly kept in mind, the rapidly growing technical literature would benefit by the absence of many papers and articles which only tend to confuse the mind of the engineer earnestly searching for solutions of problems that conform as closely to the actual conditions as is possible with the means presented by the respective state of development of science.

Thus, if, for example, the author had investigated eccentric loaded connections by the photo-elastic method, he would have come nearer the truth than by the alignment charts presented as the result of his endeavors.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

UPLIFT AND SEEPAGE UNDER DAMS ON SAND

Discussion

By MESSRS. EDWARD GODFREY, AND HIBBERT M. HILL.

EDWARD GODFREY,[19] M. AM. SOC. C. E. (by letter).[19a]—As an example of the trend of thought in all recent papers on the subject of dams (particularly as to under-pressure), this paper is significant and timely. Prior to about 1910 books on dams, at least in the English language, contained no mention of under-pressure as a factor in design. Papers on the subject that referred to under-pressure were almost nil. Nearly to the present day, the reports of the many failures that have occurred, in no case attribute the failure to under-pressure. Many such reports, however, have placed the blame for the failure on the foundation, no analysis being made of the manner of occurrence.

By its very title, Mr. Harza's paper presupposes that dams may be built on the most unstable of material, where flowing water is a possibility, namely, sand.

A significant statement is that of Conclusion 4. If under-pressure lifts a dam, it is to be expected that the water will flow through and wash out the foundation. This is what has led engineers to conclude that the foundation was at fault in the failure. Masonry and concrete dams that have been lifted and floated down stream, or have slid on their foundations, or have overturned because of insufficient base width to resist under-pressure, have not failed because of foundation weakness, but because of insufficient weight of material.

On the other hand, blow-outs below a dam follow exactly the course described by Mr. Harza in Conclusion 4: Foundation material flows out at the blow-out and gives place to other material. Even this type of failure is due to under-pressure of the water-soaked earth or sand pressing upward on the earth crust below the dam and removing it, which allows the sub-surface material to flow.

NOTE.—The paper by L. F. Harza, M. Am. Soc. C. E., was published in September, 1934, *Proceedings*. This discussion is printed in *Proceedings* in order that the views expressed may be brought before all members for further discussion.

[19] Structural Engr., Pittsburgh, Pa.

[19a] Received by the Secretary, October 5, 1934

HIBBERT M. HILL,[20] ASSOC. M. AM. SOC. C. E. (by letter).[20a]—A rational treatment of the subject of seepage under dams on permeable foundations will no doubt be greatly advanced by this paper. Without question, Mr. Harza's treatment is fundamentally correct.

It is worth noting that, for a homogeneous foundation, determination of the flow and pressure conditions within the foundation is a geometrical problem. The distribution of flow and pressure may be depicted by an orthogonal family of curves forming the flow net. In a homogeneous foundation, this net is independent of the permeability and of the head and tail-water pressures. Its form is dependent only upon that of the base of the dam and of the ground under it. There is, therefore, a simple linear relation between the flow nets of model and prototype.

For such homogeneous foundations it may be shown[21] that if the flow net is so drawn as to divide the space within the foundation into approximate squares, and that if there are N vertical squares (flow lines) and M horizontal squares (equi-pressure lines) then the flow is equal, approximately, to:

$$Q = k \frac{N}{M} h \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (15)$$

in which, N and M are determined from the model, and k and h from the prototype. As a practical matter, only the near vicinity of the structure need be considered, as the flow diminishes rapidly with distance from the structure.

Mr. Harza's values of pressure are given in decimal fractions of head. The constant value of the tail-water pressure must be added in all cases to obtain the total uplift.

Flow nets for his various experiments have been determined by the writer from observations on glass-plate models.[22] The profile of a dam is cut from a thin impervious material and placed between two plates of glass. Flow beneath an actual dam is simulated by allowing water to pass between the plates. When a dye is introduced at points along the up-stream edge of the model, the lines of flow become visible. As bearing on the geometrical properties of the flow net, it was interesting to observe that when the head in the glass-plate model was reversed, the dye retraced its former path exactly.

Mr. Harza's treatment of the escape gradient and flotation is especially worthy of study. This problem is of especial interest in the design of sand dams, in which stability of the toe must be obtained. The failure of such sloping faces (as may well be the case, also, for foundation failures), is the result of a combination of flotation and erosion by water issuing farther up the slope. The author's critical "flotation gradient" appears in such cases to be an upper limit. The writer's observations on sand models lead him to believe that there are various degrees of flotation—a progressive "quickening"

[20] Senior Engr., U. S. Engr. Office, St. Paul, Minn.

[20a] Received by the Secretary October 15, 1934.

[21] "Hydraulik", von Ph. Forchheimer, p. 81, Teubner, Berlin, 1930.

[22] *Civil Engineering*, January, 1934, p. 32.

of the sand, as the escape gradient increases. This quickening renders the sand more susceptible to movement by forces other than flotation, as by erosive forces, or gravity on a slope, so that, for particular conditions, failure may occur for values less than the author's flotation gradient. Under such circumstances, surprising results may be obtained with a filter of coarse material over the affected area. For example, a sand model, 40 cm in height, with a top width of 25 cm, and 1 on 3 slopes, began to fail progressively with a head-water of about 33 cm and tail-water of about 15 cm. A layer of $\frac{1}{4}$-in. coarse sand, placed on the down-stream slope, made the model stable even with no tail-water.

The general subject of seepage flow offers a fascinating and almost unexplored field for research, both mathematical and experimental. Various investigators have made a beginning on the subject. As Mr. Harza suggests in Conclusion 12, some organization could render valuable service by digesting and correlating the available data, and by encouraging further research.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

A GENERALIZED DEFLECTION THEORY FOR SUSPENSION BRIDGES

Discussion

BY MESSRS. STERLING JOHNSTON, HAROLD E. WESSMAN, AND C. H. GRONQUIST

STERLING JOHNSTON,[51] M. AM. SOC. C. E. (by letter).[51a]—Although this paper, as well as other highly commendable papers on suspension bridges, bears witness to the present high standards of development, there still remains a question of fundamental importance that is rarely, and then only sketchily, treated in print, namely, that of establishing an acceptable, upper limit of flexibility to which a suspension bridge may be designed. The desire of the writer, therefore, is to call attention to the need of a more uniform standard of measurement of the elastic behavior of a suspension bridge; point out the bearing that this question may have on the author's analysis of "Designs I and II"; and lay proper emphasis on its economic importance.

In Mr. Steinman's analysis of the comparative rigidity and economy of the continuous and two-hinged types of stiffening trusses, the degree of rigidity is based on the maximum deflections at the centers of the main and side spans. While this method is commonly used for making similar comparisons, it may well be questioned as to whether such a gauge is comparable for a case of this kind, in which two types of stiffening trusses of widely different deflection characteristics are being compared. Although the maximum center deflections are necessary for establishing navigable clearances, they do not, in the case of the main span, correspond with the deflections for maximum bending moment or for maximum change of grade.

Another unit of measurement, commonly used for such comparisons, is the percentage of change of grade under the live loads placed in the position or

NOTE.—The paper by D. B. Steinman, M. Am. Soc. C. E., was published in March, 1934, *Proceedings*. Discussion on this paper has appeared in *Proceedings* as follows: May, 1934, by E. Pavlo, Esq.; August, 1934, by Messrs. Jonathan Jones, A. Müllenhoff, H. Cecil Booth, Jacob Feld, and Glenn B. Woodruff, Howard C. Wood, and Ralph A. Tudor; September, 1934, by Messrs. L. J. Mensch, A. A. Eremin, Hans H. Bleich, F. H. Frankland, Gustav Lindenthal, Julian W. Shields, A. W. Fischer, and J. M. Frankland; and November, 1934, by Messrs. Fredrik Vogt, Leon S. Moisseiff, and A. Mitchell and G. T. Parkin.

[51] Engr., McClintic-Marshall Corporation, Bethlehem, Pa.

[51a] Received by the Secretary October 19, 1934.

positions, for maximum change; which simply means the change, in degrees, in the slope of the roadway due to the influence of the live load. This is the more preferable gauge which should be used in the comparisons made by the author and, in fact, should be more generally adopted for all such comparisons. If it is used as a basis of comparison for "Designs I and II," the degree of rigidity of each design should be based on the relative changes in the slopes of the main and side-span stiffening trusses at, or near, the towers rather than on the maximum deflections at the centers of the main and side spans.

The determination of the limiting unit of flexibility to which a suspension bridge may be designed is fully as important, both economically and practically, as the determination of the live loads to be carried, or the unit stresses to be used. What then is the upper limit of flexibility to which a suspension bridge may be designed, and how is it to be measured?

In considering this "unit of flexibility," care should be used to distinguish between the unit itself, and the functions of the unit. To illustrate: The unit working stress for a given steel is a function of its physical properties, factor of safety, shape, etc. Similarly, the unit that measures the degree of flexibility of a suspension bridge, is a function of the sag ratio of the cable; the depth ratio of the stiffening truss; the ratio of live to dead load; and possibly temperature changes. (The writer is inclined to believe that "temperature changes" should be considered a function of the stresses only.) While all these "functions" are necessary factors in the analysis of the deflection problem, no single one of them can be used as a gauge for a well-balanced design. They are simply a means to an end, and while the problem of how to keep the deflections within the prescribed limits is an all-important one, its consideration is beyond the scope of this discussion. The writer is primarily concerned with the unit itself, and more especially with the magnitude of the unit. In other words, the writer's query is not a question of how to stiffen a suspension bridge, but how much to stiffen it because, if "percentage of change of grade" is accepted as the basic unit with which to measure the degree of flexibility of a suspension bridge, the question naturally arises as to what this "percentage" should be.

While these two questions of how to stiffen a suspension bridge, and how much to stiffen it, are closely related, they are, nevertheless, independent questions, each of which must be approached from an entirely different angle. The question of "how" involves a highly technical analysis of the component parts of the structure, whereas the question of "how much" depends almost entirely on the nature and demands of the traffic for which the bridge is to be designed. The connecting link between "how" and "how much" is found in the effect of the dead weight of the suspended structure on its own stiffness. The technical analysis, for example, is based largely on the dead load as a controlling factor. Similarly, the degree of flexibility is a function of the dead load, the stiffness of the stiffening truss, and the sag ratio of the cable.

The practice of utilizing the dead load as a stiffening factor is of comparatively recent origin, and its economic importance is probably not yet

fully appreciated. For spans of moderate length, the degree of stiffness is largely under the control of the stiffening trusses. For spans of great length, however, the degree of stiffness, incidental to the dead load alone of a given type of construction, may be such as to more than satisfy the criteria set up as the permissible flexibility. In such a case, a lighter, cheaper, and more flexible type of construction may be justified; provided, of course, that the deflections are kept within the prescribed limits.

If, on the other hand, the degree of stiffness, incidental to the dead load alone, is inadequate; then special provision must be made for the additional stiffness required. Any such provision necessarily adds to the cost. In fact, the stiffness and cost of a suspension bridge are very closely related; other factors being comparable, the more rigid the bridge is, the more costly it will be. It follows, therefore, that both the degree of stiffness and its unit of measurement are factors of fundamental economic importance.

The writer has reached the conclusion that the required degree of stiffness for any bridge depends almost entirely upon the nature and demands of the traffic for which the bridge is to be designed. A suspension foot-bridge, for example, may be so flexible as to cause alarm to some of its passengers, and yet be perfectly safe and serviceable. On the other hand, the deflections of a modern railway bridge must be held within such narrow limits as generally to exclude the suspension type from serious consideration when placed in competition with the arch or cantilever types.

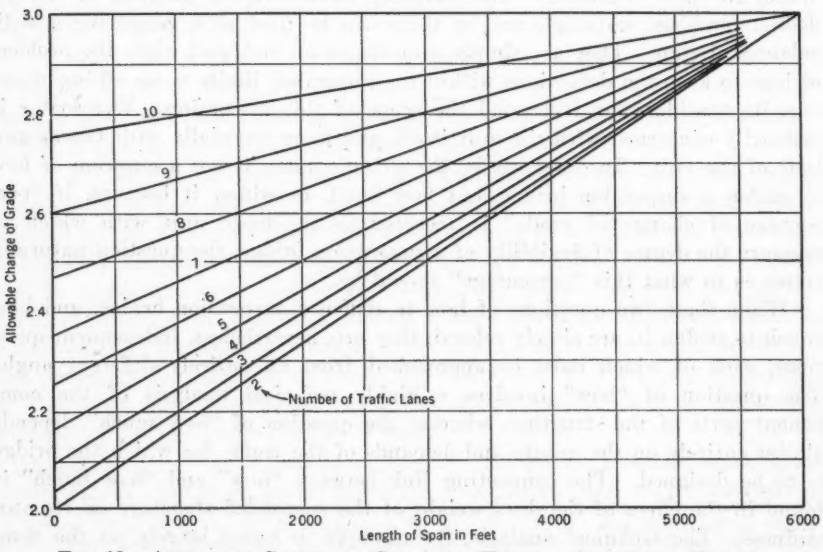

FIG. 13.—ALLOWABLE CHANGE OF GRADE FOR HIGHWAY SUSPENSION BRIDGES (FOR LIVE LOAD ONLY)

The degree of stiffness required of a highway bridge, like that of the railway type, must be based on the nature and demands of the traffic for which the bridge is to be designed. The nature of the traffic carried by these

two types of bridges, of course, is radically different. In the case of a single-track railway suspension bridge, for example, the entire moving load for which the bridge is designed comes on to the span at one time, at high speed, and with heavy impact. In the case of a large highway suspension bridge, on the other hand, the total live load for which the bridge is designed is seldom applied and even if it is, the application, as a whole, is so gradual as to cause practically no impact. Moreover, the positions of the congested live loads which produce the maximum changes of grade are such as may never occur during the life of the bridge. Therefore, while the principles of design are the same for both railway and highway types, the difference in the nature of the traffic permits of a much wider range in the computed changes of grade for the highway type.

Fig. 13 is submitted as a possible answer to the question that has been raised, as to what constitutes a permissible upper limit of flexibility for highway suspension bridges. It is based on the generally accepted thesis that an increase in the length of span, or an increase in the number of traffic lanes, permits a corresponding increase in the computed change of grade. The chart includes permissible changes of grades for live load only. It does not include temperature changes. The total grade, including temperature changes and normal grade, however, should not exceed 5 per cent.

The author is to be congratulated for this contribution to the literature on suspension bridge design. His generalized deflection theory furnishes a much needed, rational analysis of the continuous stiffening truss that has been the riddle of this type of bridge for so many years.

HAROLD E. WESSMAN,[52] ASSOC. M. AM. SOC. C. E. (by letter).[52a]—It is quite evident to any one who has studied this field that a vast amount of painstaking labor lies behind this paper. Mr. Steinman is to be highly congratulated on his new contribution to the theory of analysis of suspension structures.

After reading the literature of suspension bridges, one might conclude at first that the subject has been completely dissected and analyzed, and that nothing is left for future investigators; but when one recalls the notable contributions in recent years, it is apparent that the entire field is experiencing a critical review; that much thinking has been done recently; and, what is most important, much thinking and study still remain to be done.

Earlier concepts of the action of suspension bridges placed too much emphasis on a mistaken interpretation of the function of the stiffening truss. There was, and for that matter still is, a tendency to consider the stiffening truss as the major structural element, with the cable as a minor element added to assist the truss. This assignment of relative importance of the units was analogous to "placing the cart before the horse."

The major structural element in a suspension bridge is the cable. The stiffening truss is added to help the cable. If the cable does not need help, the stiffening truss may well be omitted, as was done on the George Washington Bridge across the Hudson River in New York City. Does the deflection

[52] Assoc. Prof. of Mechanics and Structural Eng., Univ. of Iowa, Iowa City, Iowa.
[52a] Received by the Secretary November 1, 1934.

theory apply in a case such as this? Perhaps. Even if the machinery of analysis may be the same, regardless of how one classifies cable and truss, it is important that engineers approach this problem with a rational perspective, a correct philosophy. It is well to keep in mind the valuable discussion[53] by O. H. Ammann, M. Am. Soc. C. E., of the paper entitled, "Costs of Suspension Bridges," by J. A. L. Waddell, M. Am. Soc. C. E.

With reference to the author's theory, one minor point is worth noting. His basic differential equation for the deflection, $\frac{d^2\eta}{dx^2} = -\frac{M}{EI}$, is based upon beam deflections due to flexure only. Shear deflection is ignored. The basic differential equation for beam deflection is sometimes given in the following form,

$$\frac{\partial^2\eta}{\partial x^2} = -\frac{\partial \epsilon_x}{\partial y} + \frac{\partial \gamma_{xy}}{\partial x} \quad \text{.....(113)}$$

in which, the second term represents shear effects. The first term leads to $\frac{M}{EI}$ when Poisson's ratio is ignored. Ordinarily, shear deflection in a beam is so small, relative to that caused by moment, that it is of no consequence. In a truss, however, the deformation of the web system, which is a measure of shear deflection, may be considerable for part span loadings.

Even if the shear term is omitted in the differential equation, however, the method of selecting an equivalent moment of inertia for the stiffening truss will offset error due to its omission. If the deformation of the web members is taken into account in determining truss deflection, and then an equivalent beam of uniform moment of inertia is selected that will give this same deflection, shear effect is accounted for to a certain extent. The author's viewpoint in this matter would be of some interest and value.

C. H. GRONQUIST,[54] ASSOC. M. AM. SOC. C. E. (by letter).[54a]—The deflection theory presented in this paper, for distributed loading, can be modified to include the effect of any number of concentrated loads in any span simply by adding to the general load functions, A, B_1, and B_2, the new terms for these concentrated loads. The expressions for M_o and $R_{L,R}$ will also naturally be augmented by terms contributed by the concentrated loads. Since $\frac{d^2M_o}{dx^2} = 0$ for concentrated loads, there will be no contributions to the G-functions. Thus, there will be found only one set of integration constants, C_1, C_2, for each span subjected only to concentrated loads.

The contributions of the concentrated loads to B_1 in each span will be the sum of the simple beam reactions produced by these loads. The contributions to B_2 in each span will be the difference of the reactions. In the side spans

[53] *Transactions*, Am. Soc. C. E., Vol. 91 (1927), p. 932.

[54] With Robinson & Steinman, Cons. Engrs., New York, N. Y.

[54a] Received by the Secretary November 28, 1934.

only the reaction at the tower is considered. The contributions in the several spans are:

In the Main Span:

$$B_1 = R_L + R_R = \Sigma P \ldots\ldots\ldots\ldots\ldots\ldots (114)$$

and,

$$B_2 = R_L - R_R = \Sigma P \frac{(l-2k)}{l} \ldots\ldots\ldots\ldots\ldots\ldots (115)$$

In the Left Side Span:

$$B_1 = B_2 = R_R = \Sigma P_1 \frac{k_1}{l_1} \ldots\ldots\ldots\ldots\ldots\ldots (116)$$

and, in the Right Side Span:

$$B_1 = B_2 = R_R = \Sigma P_2 \frac{k_2}{l_2} \ldots\ldots\ldots\ldots\ldots\ldots (117)$$

In Equations (114) to (117), k, k_1, and k_2, are the distances to the concentrated loads, P, P_1, and P_2, from the left end of the main span and the free end of the side spans.

The contributions of the concentrations to A in each span are:

$$A = \int_0^l M_0 \, dx = \Sigma P \frac{k}{2} (l - k) \ldots\ldots\ldots\ldots\ldots\ldots (118)$$

The contributions to B_1 do not enter into the expression for the contribution to A for concentrated loads.

The formulas for the load functions given in Equations (114) to (118) may be applied without modification to unsymmetrical three-span and multiple-span bridges as well as to symmetrical three-span bridges with or without continuity. Several formulas of the paper for the case of spans fully loaded or unloaded would be modified, however, in accordance with the contributions due to concentrated loading.

To obtain the effect of concentrated loads without any distributed loading, it is necessary merely to make p in the formulas of the paper equal to zero. The total values of the load functions, A, B_1, and B_2, are then those of the foregoing contributions of the concentrated loads.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

SECURITY FROM UNDER-SEEPAGE MASONRY DAMS ON EARTH FOUNDATIONS

Discussion

BY MESSRS. WILLIAM P. CREAGER, AND L. F. HARZA

WILLIAM P. CREAGER,[34] M. AM. SOC. C. E. (by letter)[34a].—It has been suspected for some time that Bligh's coefficients were too conservative, and the author has been successful in proving it. However, it is the writer's opinion that even Mr. Lane's coefficients are too conservative for scientifically designed structures.

Two methods are available for the determination of the correct design to insure the safety of a dam on earth foundations as regards the possibility of piping. The first is to model the dam after those of similar kinds that have stood successfully. The second consists in the direct determination of the safe escape velocity and in the design of the dam and its appurtenances to keep this velocity within proper limits. The paper deals with the first method.

Unfortunately, the data collected by the author are extremely difficult to classify properly. Only by the remotest chance could one of the dams cited resemble another closely. Unadjustable differences consist of: (1) Depth of heel cut-off; (2) depth of toe cut-off; (3) relative length of cut-offs to length of base of dam; (4) relative resistance to flow for horizontal and vertical creep lines (quite arbitrarily, the author has adopted a ratio per foot of length of 1 to 3); and (5) the character of the foundation material. The material cannot be classified as "fine sand", "medium clay", etc., with any precision, as each class has great variations.

In Table 1(*d*) the author has tabulated seventy-five dams on silt, fine sand, and medium sand. These have been plotted in Fig. 1 in the order of magnitude of the ratio of weighted-creep distance to head. Failures are noted by circles, and non-failures by dots. The author's safe values for these materials from Table 3 have also been shown. It will be noted that

NOTE.—The paper by E. W. Lane, M. Am. Soc. C. E., was published in September, 1934, *Proceedings*. This discussion is published in *Proceedings* in order that the views expressed may be brought before all members for further discussion.

[34] Hydr. Engr., Buffalo, N. Y.

[34a] Received by the Secretary September 26, 1934.

most of the failures were for very short weighted-creep-distance ratios, showing that creep distance has perhaps the preponderating effect on safety. On the other hand the great divergence of the ratio for the dams that failed shows that the creep distance is by no means the only influence.

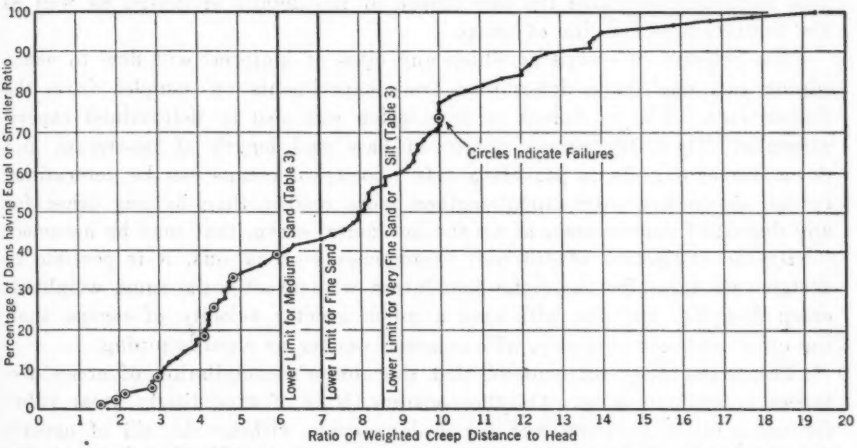
Percentage of Dams having Equal or Smaller Ratio
Ratio of Weighted Creep Distance to Head
Lower Limit for Medium Sand (Table 3)
Lower Limit for Fine Sand
Lower Limit for Very Fine Sand or Silt (Table 3)
Circles Indicate Failures

FIG. 1.—SEVENTY-FIVE DAMS ON SILT, FINE SAND, AND MEDIUM SAND, FROM TABLE 1(*d*)

Fig. 1 indicates that, for the twenty-nine dams designed with ratios less than the author's minimum recommended value of 6, twenty, or 70% of them, are still standing. Furthermore, most small earthen dams, built of fine sand without core-walls, have paths of percolation ratios of less than 6. Therefore, it would seem perfectly feasible to design for ratios less than 6, provided it is done scientifically, with full knowledge of the action of water seeping under dams. Since piping begins below the dam it will not occur if it can be prevented at that point. Therefore, it is only the velocity of the vertical seepage below the dam in which the designer is interested.

The results of electro-hydro-dynamical experiments by Professor N. P. N. Puzyrevsky,[35] which the writer was privileged to see in Moscow, tests made by the writer on both small scale sand dams, and with electro-hydro-dynamical experiments, as well as tests made by L. F. Harza, M. Am. Soc. C. E., with the hydraulic-electric analogy tray,[36] all show a crowding of the flow and very high velocities around any projections in the base or foundations. Where no cut-offs are provided, this zone of high velocity is concentrated at the angles of the base of the dam at the up-stream and down-stream ends. The water enters the foundation close to the up-stream face and leaves it close to the down-stream face at many times the average velocity, or the velocity that would obtain without such angles. The high velocity at the up-stream edge is of no importance, of course, as the foundation material is confined, but at the down-stream edge it is the direct cause of piping. If a cut-off is placed at the up-stream end of the base, the zone of high velocity

[35] *Engineering News-Record*, June 14, 1934, p. 761.
[36] *Proceedings*, Am. Soc. C. E., September, 1934, p. 967.

is transferred to the end of the cut-off, as shown by Mr. Harza.[37] A down-stream cut-off, which should always be used, would transfer the zone of high velocity away from the surface to the bottom of the cut-off and, if properly designed, would reduce the escape velocity to a safe amount. All this testimony indicates the importance of the details of design as well as the limitation of the line of creep.

The velocity of escape at which any class of material will flow to cause piping can readily be determined from experiments on samples from the foundations. The coefficient of percolation can also be determined experimentally. Then, the proper length of base and length of up-stream and down-stream cut-offs to provide a safe velocity of escape can be determined by the electro-hydro-dynamical method more readily than by any other for any degree of perviousness along the horizontal creep, that may be assumed.

By the utilization of different arrangements of cut-offs, it is possible to design two dams for the same foundation with exactly the same weighted-creep distance, but one will have a much greater velocity of escape than the other and, consequently, will be much weaker as regards piping.

The writer does not contend that the author's compilation of creep distances is without value. On the contrary, it is of exceedingly great value for comparative purposes and for designs made without the aid of experimentation. However, the writer does take exception to Mr. Lane's statement (under the heading "A NewMethod of Analysis") that: "In the present state of knowledge the only method of analyzing the probability of failure from flow along the creep line [piping failure] seems to be a study of the actions of actual dams."

L. F. HARZA,[38] M. AM. SOC. C. E. (by letter).[38a]—While it contains much other interesting material of practical value, this paper is, nevertheless, devoted mostly to the exposition of Mr. Lane's "weighted-creep" principle, in which horizontal creep distance is given a value of only one-third that of vertical creep. The author's approach is entirely empirical, which has led to the erroneous conclusion that the apparent lower resistance of horizontal seepage is necessarily the result of poor contact, or actual "roofing," due to a combination of stratification and a settlement of the foundation material away from the bottom of the structure.

Rational analysis by mathematics and its equivalent, the electric analogy,[39] proves conclusively that even in a homogenous foundation in perfect contact with the structure, apparent horizontal creep resistance along the base may have a value much less than one-third the average resistance to vertical seepage around the cut-offs, the actual value being related to the relative depth of cut-offs in proportion to their distances apart.

Thus, for example, in case of both heel and toe cut-offs of equal depth, d, as compared with the spacing, b, the hydraulic gradient or creep resistance per

[37] *Proceedings*, Am. Soc. C. E., September, 1934, p. 982, Fig. 10.

[38] Cons. Engr. and Pres., Harza Eng. Co., Chicago, Ill.

[38a] Received by the Secretary October, 27, 1934.

[39] "Uplift and Seepage Under Dams on Sand," by L. F. Harza, *Proceedings*, Am. Soc. C. E., September, 1934, p. 967.

unit distance along the base, as compared with the creep resistance per unit distance along the vertical faces of cut-offs (equivalent to Mr. Lane's weighting of horizontal creep), should be, as follows:

Space, b, between heel and toe cut-offs.	Approximate weighted value of horizontal creep.
$5.0d$	1/3
$2.5d$	1/4.2
$2.0d$	1/7
$1.5d$	1/9

The weighting of horizontal creep resistance by any fixed relation to the vertical is thus obviously incorrect in basic principle.

Considering the meager extent of available quantitative data, the author has done well to discover and verify empirically the fact that horizontal creep offers less resistance than vertical creep; but to try to express this ratio quantitatively and at a fixed value of one-third is going far beyond the accuracy justified by the data. Unfortunately, the relation is far too complicated to be expressed by any such simple ratio.

The foregoing comments apply to the condition of perfect contact between the bottom of the structure and the foundation material; that is, without roofing. It is granted that roofing is a condition almost inevitable in the case of a dam with smooth under-surface, founded on piles. It is also quite possible if the dam is founded directly on the earth. If there is even a strong possibility of roofing, however, horizontal creep distance should be given no weight whatever, even one-third would be an unsafe assumption. The writer believes in the use of longitudinal scoring of the sand bed ahead of the placing of concrete to furnish what might be called a "corrugated foundation" on the under side of the concrete to insure against roofing.

The effect of stratification, although not experimentally demonstrated, is believed to be opposite from that assumed by the author, unless in the case of the occurrence of an exceptionally porous stratum. It may be stated as a general principle that the rate at which seepage head dissipates will be rapid and, consequently, the hydraulic gradient will be steep in proportion to the tendency of the flow to concentrate. Stratification in general will tend to concentrate seepage flow rather than permit its diffusion.

The writer is not a believer in reliance alone upon abstract mathematics unsupported by experimental data for the solution of the problem of seepage under dams where non-homogeneity of material may so greatly affect the conclusions. He does believe, however, that mathematics and the electric analogy can be relied upon to establish fundamental principles, around which to build the experimental data, and that the problem should be approached in that manner.